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ABSTRACT

This is part two of a two-part SMSG text in algebra for students whose mathematical talents are underdeveloped. Additional drill materials are included in this text and terminology is kept to a minimum. Chapter topics include: (1) factors and exponents; (2) radicals; (3) polynomials; (4) rational expressions; (5) truth sets of open sentences; (6) truth sets and graphs of sentences in two variables; (7) systems of open sentences; (8) quadratic polynomials; and (9) functions. (MN)

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Introduction to Algebra

Student's Text, Part II

REVISED EDITION

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Chapter 11

FACTORS AND EXPONENTS

In mathematics we often run into problems where it is important to know whether a given number is divisible by another number. For example, in finding the sum of $\frac{1}{12}$, $\frac{1}{15}$, and $\frac{1}{8}$, we need to find the least common denominator. This, as you remember, is the smallest number that is divisible by 12, 15, and 8. Also in order to obtain a simpler name for a fraction such as $\frac{84}{360}$ we would be interested in finding the largest divisor of both 84 and 360. An understanding of the meaning and application of factoring is helpful in finding these numbers.

11-1. Factors and Divisibility.

In earlier work in algebra, we have used the set of real numbers and also different subsets of the reals: As examples, the set of integers and the set of rational numbers have been considered. In the work of this chapter, we shall be concerned only with the subset of the real numbers known as the positive integers. You have met this set before. A listing of it looks like this:

$$\{1, 2, 3, 4, 5, 6, \ldots\}.$$

In this chapter, the word "number" will refer to a member of this set.

Although fractions are mentioned in the following story, the story is intended to illustrate an important idea about the positive integers.

> Once upon a time, there was a farmer who had 11 cows. When he died, these 11 cows were left to his three sons. His will said that $\frac{1}{2}$ of the cows should be left to Charles, $\frac{1}{4}$ of the cows to Richard, and $\frac{1}{6}$ of the cows to Oscar. The sons argued about this, because none of them wanted just a piece of a cow, as the will seemed to require. As they were arguing, a stranger came along, leading a cow to market. The three boys told him of their problem, and the stranger said, "That's simple. Just let me give you my cow, and then try it." The boys were delighted, for they now had 12 cows instead of 11. Charles took one-half of these, or 6 cows. Richard took one-fourth of them, that is, 3 cows. Oscar took one-sixth of them, or 2 cows. The 11 cows which the farmer had willed were now happily divided. The stranger took his own cow and went on his way.

The outcome of this story may cause you to think that the boys did not get their fair shares. But notice that each boy actually got more, since it is true that

$$6 > \frac{11}{2}, \quad 3 > \frac{11}{4}, \quad \text{and} \quad 2 > \frac{11}{6}.$$

Even so, however, there seems to be something "fishy" about the story. What made such an unusual solution possible?

For some reason, the people of the story found it much easier to deal with 12 cows than with 11. The reason is that 12 can be divided by each of the numbers 2, 4, and 6, with remainder zero. The number 11, however, cannot be divided by any of these numbers, with remainder zero. Do you see how this made 12 a more convenient number in the story?

Actually, the phrase "divided by, with remainder zero" points directly to a very important mathematical idea, which gives us another choice of words. Instead of saying "12 is divisible by 6, with remainder zero," we can say that "6 is a factor of 12".

Before considering a definition of the word "factor," study the following examples of its use and see if you can answer the questions asked:

4 is a factor of 12, because $4 \times 3 = 12$.

5 is not a factor of 12, because 12 is not a multiple of 5.

Is 3 a factor of 12? Why or why not?

Is 2 a factor of 12? Why or why not?

Is 7 a factor of 12? Why or why not?

Notice that saying a number is a factor of 12 means that 12 is a positive multiple of that number. (A multiple of a number is the product of that number and an element of the set of integers.) For example,

4 is a factor of 12.
12 is a positive multiple of 4. $(12 = 4 \cdot 3)$

5 is not a factor of 12.
12 is not a multiple of 5.

We can now turn our attention from specified numbers to variables and make the following statement:

> To say that a positive integer x is a factor of a positive integer y means that y is a positive multiple of x.

Any positive integer has itself and the number one as factors. For instance, 12 and 1 are factors of 12. 12 is a non-zero multiple of 12 and of 1. The other factors of 12--the numbers 2, 3, 4, and 6--are called proper factors of 12.

By way of contrast, the number 11 does not have any proper factors. The only factors of 11 are 11 and 1.

These examples illustrate the following definition of a proper factor:

> A positive integer m is a proper factor of the positive integer n if $mq = n$, where q is some positive integer other than n and 1.

In order to test your understanding of this definition, try using the definition to show why 3 is a proper factor of 12 and to show why 12 is not a proper factor of 12. In each case, what are the numbers m, n, and q?

Check Your Reading

1. Is 12 a multiple of 6? of 4?
2. Is 6 a factor of 12? Is 4?
3. Is 12 a proper factor of 12? Is 2?
4. Does 11 have any proper factors?
5. What is a factor of a number?
6. What is a proper factor of a number?
7. What do we mean when we say that a positive integer m is a proper factor of a positive integer n?

Oral Exercises 11-1a

1. In answering each of the following questions, give a reason for your answer as the examples show.

   Example 1. Is 5 a factor of 45? Yes, since $5 \times 9 = 45$.

   Example 2. Is 5 a factor of 46? No, since 46 is not a multiple of 5. There is no integer q such that $5q = 46$.

   (a) Is 3 a factor of 24?

   (b) Is 5 a factor of 24?

   (c) Is 9 a factor of 24?

   (d) Is 24 a factor of 24?

   (e) Is 36 a proper factor of 36?

   (f) Is 1 a factor of 5?

2. List the set of multiples of 6.

3. List the set of multiples of 12.

4. List the set of all factors of 12.

5. List the set of proper factors of 12.

6. If $mn = p$, where m, n, and p are positive integers, is m a factor of p? Is n a factor of p?

7. Since 3 is a factor of 18, is $\frac{18}{3}$ also a factor of 18?

8. If m is a factor of p, is $\frac{p}{m}$ a factor of p? Is it a proper factor?

9. If we know one factor of a number, how can we immediately find another?

Problem Set 11-1a

Answer each of the following questions. Give a reason for your answer, as shown in the oral exercises.

1. Is 13 a factor of 91?

Problem Set 11-1a
(continued)

2. Is 30 a factor of 510?

3. Is 12 a factor of 204?

4. Is 10 a factor of 100,000?

5. Is 6 a factor of 151,821?

6. If 3 is a factor of 51, what is another proper factor?

7. If 3 is a factor of 57, what is another proper factor?

8. If 5 is a factor of 65, what is another proper factor?

9. If 7 is a factor of 161, what is another proper factor?

10. If 29 is a factor of 87, what is another proper factor?

11. If 23 is a factor of 437, what is another proper factor?

12. If a positive integer c is a proper factor of a positive integer q, what is another proper factor?

13. List the set of all factors of 18.

14. List the set of all proper factors of 18.

15. List the set of positive multiples of 18.

16. Write each of the following numbers as a product of proper factors, if it is possible. It will not be possible in every case, because some of the numbers have no proper factors.

Example: $24 = 2 \times 12$

or $24 = 3 \times 8$

There are other possibilities.

(a) 85
(b) 51
(c) 52
(d) 29
(e) 93
(f) 92
(g) 37
(h) 94
(i) 55
(j) 61
(k) 23
(l) 123
(m) 57
(n) 65
(o) 122
(p) 68
(q) 95
(r) 129
(s) 141
(t) 101

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Is 3 a factor of 151,821?
Is 6 a factor of 151,821?
Is 12 a factor of 187,326,648?

These are examples of questions that can be answered on the basis of the definition of factor given in the previous section. You can see, however, that answering them would involve some time-consuming arithmetic, such as dividing 187,326,648 by 12. In much of our future work in algebra, it will be convenient to have a shorter way of deciding whether or not one number is a factor of another number. In this section, we shall uncover some of these methods.

Consider first the set of even positive integers:

$\{2, 4, 6, 8, 10, 12, 14, 16, 18, 20, \ldots\}$.

2 is a factor of every number in this set; in fact, an integer with factor 2 is exactly what is meant by an even integer. Notice that every even number has a common name whose last digit is one of the following: 0, 2, 4, 6, 8. Already then we have an easy way of deciding whether or not 2 is a factor of a number.

The set of positive multiples of 5 can be listed like this:

$\{5, 10, 15, 20, 25, 30, 35, 40, \ldots\}$.

Is it possible to look at the last digit of a numeral and tell whether or not the number it names has a factor of 5?

The set of positive multiples of 10 is indicated below:

$\{10, 20, 30, 40, 50, 60, 70, 80, \ldots\}$.

Here again, do you see that there is a way of deciding from the numeral whether or not the number has a factor 10?

It seems fairly simple to look at a numeral and decide about the factors 2, 5, and 10. It is not quite so simple to make a decision about the factor 3, but the table below should help you to see a method. Try to find a connection between the sum of the digits and the factor 3. (Some of the table has been left for you to complete on a separate sheet of paper.)

Number	Sum of Digits in Numeral	Is 3 a factor of the number?
147	1 + 4 + 7 = 12	yes (3 · 49 = 147)
36	3 + 6 = 9	yes (3 · 12 = 36)
49	4 + 9 = 13	no
84		
112		
562		
23,412		

If you had trouble detecting any of these methods or in putting them into words, the following summary may help. It gives the methods for deciding whether or not 2, 3, 5, and 10 are factors of a given number; in each case, the "test" is based on the common decimal numeral for the number.

> If the last digit is a multiple of 2, the number has a factor 2.
>
> If the sum of the digits is a multiple of 3, the number has a factor 3.
>
> If the last digit is either "0" or "5", the number has a factor 5.
>
> If the last digit is "0", the number has a factor 10.

Because of the relationship between factors and dividing, the four tests above are also called tests for divisibility. In the problem set, you will have a chance to discover some other tests for divisiblity.

Problem Set 11-1b

1. Which of the following numbers are divisible by 2? By 3? By 5? By 10?

    (a) 207
    (b) 3894
    (c) 14142
    (d) 44726
    (e) 47200
    (f) 379242
    (g) 3825
    (h) 21460
    (i) 66248
    (j) 23824
    (k) 46008
    (l) 739200

Problem Set 11-1b
(continued)

2. If you find that 2 is a factor of a number and also that 3 is a factor, would 6 then be a factor? Hint: An open phrase for a number that has 2 and 3 as factors could be $2 \cdot 3 \cdot n$. Is 6 a factor of $2 \cdot 3 \cdot n$? Why?

3. Use the test discussed in Problem 2 above to find which of the following numbers has a factor of 6.(Do not divide.)

   (a) 227160
   (b) 821734
   (c) 1031514
   (d) 34125
   (e) 12318
   (f) 32142
   (g) 999360
   (h) 213405
   (i) 27342
   (j) 132264

4. How could you test numbers for divisibility by 15? (Problem 2 should help you if you have trouble.)

5. Use your test for divisibility by 15 on each of the numbers in question 3. *Do not* divide.

*6. You remember that the divisibility test for 3 uses the sum of the digits. There is a divisibility test for 9 that also uses the sum of the digits. See if you can discover and state this divisibility test for 9.

*7. In the following list of numbers, which ones are divisible by 4?

   2 14 24 16 28 304
   402 314 124 216 528 708
   702 1014 724 916 3028 1004

   Do you see any connection between divisibility of a number by 4 and the *last two digits* of the numeral? See if you can state a divisibility test for 4.

*8. Try to state a test for divisibility by 12. Use your test on parts (b), (c), (e), (k), of problem 1, and check by division.

*9. There is a divisibility test for 8, very much like the test for 4. See if you can discover and state a divisibility test for 8.

Problem Set 11-1b
(continued)

*10. There is a divisibility test for 11. Study each of the following examples.

495	$(5 + 4) - (9) = 0$	$495 = (11)(45)$
611325	$(6+1+2) - (1+3+5) = 0$	$611325 = (11)(55575)$
9240	$(9 + 4) - (2 + 0) = 11$	$9240 = (11)(840)$
81917	$(8+9+7) - (1 + 1) = 22$	$81917 = (11)(7447)$
9097	$(?) - (?) = ?$	$9097 = (11)(?)$
121	$(?) - (?) = ?$	Is 11 a factor of 121?
99902	$(?) - (?) = ?$	Is 11 a factor of 99902?

Complete this sentence, after studying the above examples:

> If the difference of the sums of sets of alternate digits in the common name of a number is a multiple of ____ , then the number itself is a multiple of ____ .

*11. There is a test for divisibility by 7. Consider the number 13762. Take twice the last digit and subtract it from the number represented by the remaining digits. Is the remainder divisible by 7?

$13762 ;\quad 1376 - 2(2) = 1372$

Continue the same pattern. Is the remainder always a multiple of 7?

$1372 ;\quad 137 - 2(2) = 133$
$133 ;\quad 13 - 2(3) = 7$

Consider some other examples.

$6538 ;\quad 653 - 2(8) = 637$
$637 ;\quad 63 - 2(7) = 49$
$49 ;\quad 4 - 2(9) = -14$

$9814 ;\quad 981 - 2(4) = 973$
$973 ;\quad 97 - 2(3) = 91$
$91 ;\quad 9 - 2(1) = 7$

Problem Set 11-1b
(continued)

Complete the sentence:

> If the difference between _______ the number represented by the last digit and the number represented by the __________ digits is a multiple of seven then the number itself is a multiple of seven.

Is this test a useful test? Why?

11-2. Prime Numbers and Prime Factorization.

In working with factors of numbers, we often choose the factors from a set called the set of prime numbers. The set of prime numbers is a subset of the set of positive integers. The definition of a prime number will be considered in the following problem set and in the next section of the text.

First, however, let's discuss certain ways in which a set of numbers might be determined, or generated. Suppose we start with the number one, then add one, add one again, add one again, and so on, without end. The following set of numbers would be generated.

$$\{1, 2, 3, 4, 5, 6, 7, 8, \ldots\}.$$

This, of course, is the familiar set of counting numbers. It is correct to say then that the set of counting numbers is generated by the number one and the operation of addition. That is, we can build the set of counting numbers by starting with one number--1--and with one operation--addition.

What set of numbers is generated by the number 2 and the operation of addition? Starting with the number 2, we add 2, getting 4, add 2 again, obtaining 6, and so on. The result is the set

$$\{2, 4, 6, 8, 10, 12, 14, 16, \ldots\},$$

the set of even positive integers.

In the problem set, you will be asked to generate sets of numbers by using a specified number and a specified operation.

Problem Set 11-2a

1. What set of numbers is generated by the number 3 and the operation of addition? Does every number in this set have the proper factor 3?

2. How could the set of all positive integers with factor 5 be generated? List the set.

3. How could the set of all positive integers with factor 7 be generated? List the set.

4. Write a list of all the positive integers from 1 to 100. Then follow the instructions given below. Do this work carefully, because it will be used in the next section to decide what a prime number is.

   (a) In the list, cross out every number that has 2 as a proper factor. Do not cross out the number 2 itself since 2 is not a proper factor of 2. Beneath each number that you cross out, write "2", to show that 2 is a proper factor of the number.

   (b) After 2, the next number in the list is 3. Cross out every number in the list (that has not already been crossed out) that has a proper factor of 3. Write "3" beneath each number that you cross out.

   (c) The next number in the list is 4, but 4 has been crossed out (it has 2 as a proper factor). So move on to 5, which has not been crossed out. Cross out every remaining number in the list that has a proper factor of 5, writing "5" beneath each one.

   (d) Continue this process until you can go no further.

   (e) When you have finished, some of the numbers (like 2, 3, and 5) will not be crossed out. Notice that the numbers not crossed out are numbers that do not have any proper factors.

Problem 4 in the problem set above was included for an important reason. It helps in understanding the definition of a prime number. If you followed the instructions carefully, you found that the following numbers were not crossed out:

1, 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37,
41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97.

Notice that none of these numbers have any proper factors. For this reason, they are called prime numbers, with the exception of the number one, which is not considered to be prime. In fact, the following definition can be made:

> A prime number is an integer greater than one that has no proper factors.

So we have uncovered twenty-five prime numbers (listed above, with the number 1 removed). We could have found many more if the list of numbers we started with had gone beyond 100. For example, 101 is a prime number. Why?

Because the list of numbers we started with in Problem 4 served to sift out the prime numbers, it is sometimes called a "sieve", and we shall refer to it by that name on the next few pages. Look at the number 63 in the sieve (it was crossed out) and see if you can follow these steps:

Number	Smallest Prime Factor (from sieve)		
63	3	63 3	In the sieve, the number 3 is written below 63, indicating that 3 is the smallest prime number that is a factor of 63. If 63 is divided by 3, the quotient is 21. $63 = 3 \cdot 21$.
21	3	21 7	Beneath the number 21, the number 3 is also indicated. 3 is the smallest prime factor of 21. If 21 is divided by this factor 3, the quotient is 7. $21 = 3 \cdot 7$.
7	7	7	The number 7 has not been crossed out. It is a prime number, and its smallest prime factor is 7 itself. The only other factor of 7 is 1, not a prime.

The work above can also be shown like this:

$63 = 3 \cdot 21$	3 is the smallest prime factor of 63. 21 is a factor of 63, but it is not prime.
$= 3 \cdot 3 \cdot 7$	The number 21 has now been factored. 3 is also the smallest prime factor of 21. 7 is also a prime factor.

The product $(3)(3)(7)$ is equal to 63. Thus, we can say that we have "factored" 63; that is, we have listed factors whose product is 63. Furthermore, each of the factors in this case is a prime number. So we say that we have shown a prime factorization of 63.

By way of contrast, note that 63 might be factored in the following way: $63 = (7)(9)$. However, this is not a prime factorization. Why not?

In order to clarify your thinking about prime factorization, another example is given below.

Example. Give the prime factorization of 60.

$60 = (2)(30)$	2 is the smallest prime factor of 60.
$= (2)(2)(15)$	2 is the smallest prime factor of 30.
$= (2)(2)(3)(5)$	3 is the smallest prime factor of 15. Note that all of the factors are now prime.

In giving a prime factorization of a number, there is no special order in which the factors must be listed. For example, the prime factorization of 63 might be written as $(7)(3)(3)$ instead of $(3)(3)(7)$. Although the order of the factors may be changed, there is only one selection of numbers in any prime factorization. Hence, it is correct to speak of the prime factorization of a number. Another way of saying this is to say that

> if a positive integer has a prime factorization,
> this factorization is unique.

It may seem obvious to you that each positive integer which can be factored has only one prime factorization. However, unique prime factorization is a property of the positive integers. It is not a property of all sets of numbers. For example, in the set of rational numbers, there is no unique prime factorization.

Check Your Reading

1. What is a prime number?
2. Which of the following are prime numbers: 19, 29, 39, 49, 59, 69, 79, 89, 99? Are all numbers ending in 9 prime numbers?
3. Which of these are prime numbers: 3, 13, 23, 33, 43, 53? Are all numbers ending in 3 prime numbers?
4. Are all odd numbers prime numbers?
5. Are all prime numbers odd?
6. What is the prime factorization of 63? of 60?
7. How do we start the prime factorization of a number?

Oral Exercises 11-2b

1. Give the prime factorization of each of the following numbers.

   (a) 4
   (b) 6
   (c) 15
   (d) 18
   (e) 35
   (f) 28
   (g) 27
   (h) 99
   (i) 46
   (j) 51
   (k) 24
   (l) 91
   (m) 48
   (n) 26
   (o) 20
   (p) 75
   (q) 66
   (r) 108
   (s) 144
   (t) 225

2. What is the next prime after 100?
3. In each of the following, the prime factorization of a number is given. Give a common name for the number.

   (a) $2 \times 2 \times 2$
   (b) $2 \times 2 \times 3 \times 3$
   (c) $2 \times 2 \times 5$
   (d) $2 \times 7 \times 7$

Oral Exercises 11-2b
(continued)

(e) 2 × 3 × 3 × 3
(f) 7 × 7 × 11
(g) 5 × 7 × 11
(h) 3 × 3 × 5
(i) 3 × 5 × 7
(j) 13 × 19 (Does the distributive property help? Try 13(20 - 1).)

Problem Set 11-2b

1. Tell which of the following are prime numbers. (Hint: remember how to test for divisibility by 2, 3, 5, 11.)

(a) 49
(b) 53
(c) 37
(d) 105
(e) 111
(f) 121
(g) 97
(h) 10101
(i) 9999

2. What is the largest even prime number? How many even prime numbers are there?

*3. Do you think there is a largest prime number? Why?

4. Give the prime factorization for the following numbers.

(a) 65
(b) 51
(c) 91
(d) 78
(e) 102
(f) 132
(g) 145
(h) 256
(i) 243
(j) 625
(k) 1104
(l) 732
(m) 10101
(n) 999

5. Each of the following is a prime factorization of a number. Give a common name for the number.

(a) 11 × 11
(b) 11 × 13
(c) 11 × 19
(d) 7 × 19
(e) 7 × 23
(f) 3 × 7 × 11
(g) 3 × 13 × 19
(h) 3 × 3 × 3 × 3
(i) 5 × 19 × 31

In the previous section, we considered the idea of factoring a number; specifically, we worked with the prime factorization of numbers. The numbers we worked with at that time were relatively small. In this section, let's look at a larger number; 3528, for example, and determine its prime factorization.

Number	Smallest Prime Factor	
3528	2	2 divides 3528. The quotient is 1764. 3528 = (2)(1764).
1764	2	2 divides 1764. The quotient is 882. 1764 = (2)(882).
882	2	2 divides 882. The quotient is 441. 882 = (2)(441).
441	3	2 does not divide 441, but 3 does. The quotient is 147. 441 = (3)(147).
147	3	3 divides 147. The quotient is 49. 147 = (3)(49).
49	7	3 does not divide 49; neither does 5. 7 divides 49. 49 = (7)(7).
7	7	7 is a prime number.

The steps above show that the prime factorization of 3528 is

$$2 \times 2 \times 2 \times 3 \times 3 \times 7 \times 7.$$

This expression is usually written more briefly as follows:

$$2^3 \times 3^2 \times 7^2.$$

The new symbols in this expression are explained below.

> "3^2" is read "3 squared" or "3 to the second power". It means "3 used as a factor two times". $3^2 = 3 \cdot 3$. We also say that 3^2 is a power of 3.
>
> "2^3" is read "2 cubed" or "2 to the third power". It means "2 used as a factor

three times". $2^3 = 2 \cdot 2 \cdot 2$.
2^3 is the third power of 2.

As another example of this kind of symbol, consider the symbol "5^4".

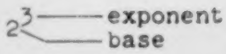
"5^4" is read "5 to the fourth power".
It means "5 used as a factor four times".
$5^4 = 5 \cdot 5 \cdot 5 \cdot 5$. 5^4 is a power of 5.

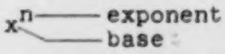
This kind of symbol can be defined easily by use of variables. For example,

"x^n" is read "x to the n^{th} power". It means "the number x used as a factor n times". $x^n = \underbrace{x \cdot x \cdot x \text{ .... } x}_{n \text{ factors}}$.

x^n is the n^{th} power of x.

Certain labels are usually attached to the two parts of this kind of symbol. These names are shown below, and will be used hereafter.





It can be said then that an exponent shows how many times the base is to be used as a factor.

From the ideas presented in this section, it should be clear that there is a strong connection between divisibility and prime factorization. To show this, let's consider the question of whether or not 252, for example, is divisible by 84. More simply the question could be worded:

"Does 84 divide 252?"

We can, of course, get the answer directly by division. Since $252 \div 84 = 3$ with remainder zero, the answer is yes.

However, suppose we look at the problem in another way, using prime factorization. We see that

$252 = 2^2 \cdot 3^2 \cdot 7$

and $84 = 2^2 \cdot 3 \cdot 7$.

What do you notice about the two prime factorizations? We observe that the prime factorization of 84 is, in a sense, completely included in the prime factorization of 252. Another way of saying this is that each prime factor of 84 is also a prime factor of 252, and the exponent of each factor of 84 is less than or equal to the exponent of the corresponding factor of 252. In this connection we can think of any number appearing without an exponent as having an implied exponent of 1. Thus $7 = 7^1$ and $3 = 3^1$, etc.

Consider another example. Does 36 divide 180? The prime factorizations are:

$$180 = 2^2 \cdot 3^2 \cdot 5$$

and

$$36 = 2^2 \cdot 3^2.$$

What is your answer?

This idea can really be thought of as an application of the multiplication property of 1. In other words, we see that

$$\begin{aligned}\frac{2^2 \cdot 3^2 \cdot 5}{2^2 \cdot 3^2} &= \left(\frac{2^2}{2^2}\right)\left(\frac{3^2}{3^2}\right)(5)\\ &= (1)(1)(5)\\ &= 5\end{aligned}$$

Check Your Reading

1. What is the prime factorization of 3528? How can we write this more briefly?
2. In the numeral "3^2", what is the number 3 called? What is the number 2 called? What number does 3^2 represent?
3. How do we read "5^4"? What does it mean?
4. What is an exponent?
5. What does "x^n" mean? How do we read it? What is the number x called? What is the number n called?
6. Why does $(2^2 \cdot 3^2)$ divide $(2^2 \cdot 3^2 \cdot 5)$?

Oral Exercises 11-2c

1. Give common names for the following powers.

   (a) 5^2
   (b) 6^2
   (c) 2^3
   (d) 3^3
   (e) 7^2
   (f) 10^3
   (g) 12^2
   (h) 5^3
   (i) 2^4
   (j) 3^5
   (k) 2^5
   (l) 5^4

2. What is the smallest prime factor of each of the following numbers?

   (a) 115
   (b) 135
   (c) 321
   (d) 484
   (e) 539
   (f) 121
   (g) 723
   (h) 125
   (i) 343
   (j) 1111112
   (k) 372463
   (l) 23232231

3. State the prime factorization of these numbers, using exponents.

   (a) 20
   (b) 16
   (c) 49
   (d) 100
   (e) 75
   (f) 18
   (g) 144
   (h) 68
   (i) 50
   (j) 27

4. For which of the following pairs of numbers is it true that the first number divides the second? For which is it false?

   (a) $2^2 \cdot 3^3$, $2^3 \cdot 3^3$
   (b) $2 \cdot 3^2$, $3^2 \cdot 5$
   (c) $3 \cdot 5 \cdot 7$, $3^2 \cdot 5^3$
   (d) $3 \cdot 5^2 \cdot 11$, $3^2 \cdot 5^2 \cdot 11^2$
   (e) $2 \cdot 3^2$, 10625
   (f) $5 \cdot 7^2$, 392

Problem Set 11-2c

1. Each of the following is a power of a prime number. Tell what power of the prime it is. (Example: $9 = 3^2$.)

   (a) 8
   (b) 16
   (c) 49
   (d) 125
   (e) 81
   (f) 32
   (g) 27
   (h) 243
   (i) 121
   (j) 169
   (k) 625
   (l) 343

Problem Set 11-2c
(continued)

2. Find the prime factorization of each of the following numbers. Use exponents in writing your answer.

   (a) 98
   (b) 432
   (c) 258
   (d) 180
   (e) 1024
   (f) 378
   (g) 729
   (h) 825
   (i) 576
   (j) 1098
   (k) 486
   (l) 3375

3. For which of the following pairs of numbers is it true that the first number divides the second? For which is it false?

   (a) $3^2 \cdot 5, \; 3^2 \cdot 5^2 \cdot 7$
   (b) $2^3 \cdot 5, \; 2^3 \cdot 7$
   (c) $2 \cdot 3 \cdot 5, \; 2^2 \cdot 3^2 \cdot 5^2$
   (d) $2^8, \; 3^8$
   (e) $5^2 \cdot 7, \; 3156$
   (f) $2 \cdot 3^2, \; 162$

4. Use prime factorization to decide whether or not the first number divides the second. Consider the variables as primes.

   (a) 36, 396
   (b) 30, 875
   (c) 16, 192
   (d) $27a^2b, \; 81ab^2$
   (e) $180a^3b, \; 1260a^4b^2$
   (f) $175, \; 162372c$

5. Find a common name for the number represented by each of the following expressions for the given value of the variable.

   (a) x^5 if x is 2
   (b) 2^x if x is 4
   (c) a^3 if a is 7
   (d) 3^a if a is 3
   (e) a^3b^2 if a is 2 and b is 4
   (f) $(a^3)(a^2)$ if a is 3
   (g) $\frac{x^5}{x^2}$ if x is 5
   (h) $(2^x)(2^y)$ if x is 3 and y is 2
   (i) m^2n^2 if m is 3 and n is 5
   (j) $(mn)^2$ if m is 3 and n is 5
   (k) $\frac{a^2b^2}{ab}$ if a is 3 and b is 4

Problem Set 11-2c
(continued)

(l) $\frac{3^x}{3^y}$ if x is 5 and y is 3

(m) $3a^4$ if a is 2

(n) $(3a)^4$ if a is 2

(o) $(5m)^3$ if m is 3

(p) 5^3m^3 if m is 3

(q) $5m^3$ if m is 3

(r) $(a + b)^2$ if a is 2 and b is 3

(s) $a^2 + b^2$ if a is 2 and b is 3

(t) $a^2 + 2ab + b^2$ if a is 2 and b is 3

(u) $(a + b)(a - b)$ if a is 2 and b is 3

(v) $a^2 - b^2$ if a is 2 and b is 3

(w) $(a - b)(a - b)$ if a is 2 and b is 3

(x) $a^2 - 2ab + b^2$ if a is 2 and b is 3

6. If the length of each side of a square is 5 inches, what is its area in square inches?

7. If the length of each edge of a cube is 5 inches, what is its volume in cubic inches?

8. If you did Problems 6 and 7 correctly, you should have stated the numbers represented by 5^2 and 5^3 for the two answers. Can you see why the words "squared" and "cubed" are used?

At the beginning of this chapter, it was pointed out that our concern would be only with the set of positive integers. Now that we have done some work in factoring, we are in a position to see that some such restriction must be made in order to make any definite statements at all concerning factorization.

For example, without our restriction, we would find it more difficult to define a prime number. If rational numbers, for example, were admitted to the discussion, the prime number 7 would certainly have factors other than itself and one, since $7 = \frac{1}{2} \times 14$. We would not have unique factorization of any kind.

You can see then that the restriction to the positive integers is basic to this chapter. Later, we may find uses for factors outside the set of positive integers. But, for the present, we shall continue to confine our attention to this set.

11-3. Factors and Sums.

Are there two numbers whose product is 72 and whose sum is 22?

There are two reasons for considering this type of question at this time. It is a question that will arise in future work, and it is a question which is directly related to the ideas on factoring which we are now studying.

One way to approach the solution is to guess. 20 and 2 are two numbers whose sum is 22; unfortunately their product is not 72. The sum of 12 and 10 is 22, but their product is not 72. Although a correct guess is always a possibility, blind guessing is a wasteful approach to the problem.

A more systematic approach might begin with the prime factorization of 72. (Remember that the product of the numbers is to be 72.)

$$\begin{aligned} 72 &= 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \\ &= 2^3 \cdot 3^2 \end{aligned}$$

We must also recall at this point that 1 is a factor of every positive integer. In our study of factors, we have usually omitted the number 1 from consideration since the product of 1 and any other number is equal to that number. However, we are now concerned with a sum as well as a product. Hence the number 1 should be included in forming our lists of possible factors. Thus for convenience we may write

$$72 = 1 \cdot 2^3 \cdot 3^2.$$

In determining the two numbers, we can place the factors in any way we choose to get a product of 72, just as long as we are sure that all of the prime factors of 72 are there. We must try different placements until we get an arrangement that also gives two numbers with a sum of 22.

One possible placement of the factors is like this:

$\underline{1 \cdot 2^3}$ and $\underline{3^2}$

Do you see that all of the prime factors of 72 appear? The numerals indicated name the numbers 8 and 9. Of course their product is 72, but their sum is not 22. They do not represent a solution to the problem.

Some other possibilities (not all of them) are listed and discussed below:

$\underline{1 \cdot 2}$	and	$\underline{2^2 \cdot 3^2}$	The numbers here named are 2 and 36. Their sum is not 22.
$\underline{1}$	and	$\underline{2^3 \cdot 3^2}$	The sum of 1 and 72 is not 22.
$\underline{1 \cdot 2 \cdot 3}$	and	$\underline{2^2 \cdot 3}$	The sum of 6 and 12 is not 22.
$\underline{1 \cdot 2 \cdot 3^2}$	and	$\underline{2^2}$	The sum of 18 and 4 is 22.

Therefore, 18 and 4 are the numbers required.

As a second example, consider the following. The sum of two numbers is 32. The product is 156. Find the numbers. As before, we first form the prime factorization of 156. Then, as before, write in the number 1. That is, we note that

$$156 = 1 \cdot 2^2 \cdot 3 \cdot 13.$$

Some possible choices for the sum are:

$\underline{1}$	and	$\underline{2^2 \cdot 3 \cdot 13}$	$1 + 156 = 157$
$\underline{1 \cdot 2}$	and	$\underline{2 \cdot 3 \cdot 13}$	$2 + 78 = 80$
$\underline{1 \cdot 2^2}$	and	$\underline{3 \cdot 13}$	$4 + 39 = 43$
$\underline{1 \cdot 2 \cdot 3}$	and	$\underline{2 \cdot 13}$	?

What can you say about the last choice? Are there others?

There was still some guesswork in seeking the solutions above. However, the guesses came after the prime factorizations of 72 and 156 were obtained, and involved the proper placement of factors into two groups.

Problem Set 11-3a

1. List four of the different arrangements of the prime factors of 72 into two groups. (Remember, one and the number itself are factors of every number.) Find the arrangement, if any, which results in a sum of 27.

2. Write the prime factorization of the product in each of the following and use it to find two numbers whose sum and product are indicated below.

   (a) The product is 18 and the sum is 9.
   (b) The product is 18 and the sum is 11.
   (c) The product is 32 and the sum is 12.
   (d) The product is 49 and the sum is 50.
   (e) The product is 54 and the sum is 15.
   (f) The product is 14 and the sum is 15.
   (g) The product is 200 and the sum is 30.
   (h) The product is 225 and the sum is 30.
   (i) The product is 56 and the sum is 56.
   (j) The product is 216 and the sum is 30.
   *(k) The product is 972 and the sum is 247.
   *(l) The product is -180 and the sum is 3. (Hint: write -180 as (-1)(180) and factor 180.)

3. Write the prime factorization of the first number in each of the following and use it to find two numbers whose product and whose sum are as indicated below.

   (a) Product is 48 and sum is 14
   (b) Product is 48 and sum is 26
   (c) Product is 48 and sum is 20
   (d) Product is 48 and sum is 19
   (e) Product is 150 and sum is 36
   (f) Product is 150 and sum is 25
   (g) Product is 150 and sum is 31
   (h) Product is 150 and sum is 30

Problem Set 11-3a
(continued)

(i) Product is 150 and sum is 151
(j) Product is 288 and sum is 34
(k) Product is 330 and sum is 37

4. Half the perimeter of a rectangular field is 34 feet and the area is 225 square feet. If the length and width are integers, find them.

5. The perimeter of a rectangle is 58 feet and its area is 54 square feet. Find the length and width.

6. The sum of the base and height of a parallelogram is 15 inches and the area is 36 square inches. What are the base and height?

7. The combined length of the base and altitude of a triangle is 62 inches while the area is 300 square inches. What are these two dimensions?

8. Find two integers, x and y, such that:

(a) $xy = 48$ and $x + y = 26$
(b) $xy = 60$ and $x + y = 19$
(c) $xy = 60$ and $x + y = 24$
(d) $xy = 700$ and $x + y = 53$

11-4. Laws of Exponents.

In a previous section the meaning of x^n was given as

$$\underbrace{x \cdot x \cdot x \quad \dots \quad \cdot x}_{n \text{ factors}}.$$

There were also some problems which may have caused you to think that indicated products such as $a^2 \cdot a^3$ and $a^x \cdot a^y$ might be written in simpler form. Let's study the first of these and see if it will help us with the second. We know that

$$a^2 = a \cdot a, \qquad a^3 = a \cdot a \cdot a.$$

So, $a^2 \cdot a^3 = (a \cdot a) \cdot (a \cdot a \cdot a)$.

Since only the operation of multiplication is involved, the parentheses are not necessary, and the sentence above may be written

$$a^2 \cdot a^3 = a \cdot a \cdot a \cdot a \cdot a.$$

The phrase on the right side of this sentence is simply a^5. So we have

$$a^2 \cdot a^3 = a^5.$$

This example, together with several others, has been placed in the table below. Try to determine a pattern for finding a simpler numeral.

Indicated Product	Meaning	Simpler Numeral	Exponents in Indicated Product	Exponents in Simpler Numeral
$a^2 \cdot a^3$	$\underbrace{a \cdot a}_{\text{2 factors}} \cdot \underbrace{a \cdot a \cdot a}_{\text{3 factors}}$	a^5	2, 3	5
$2^4 \cdot 2^2$	$\underbrace{2 \cdot 2 \cdot 2 \cdot 2}_{\text{4 factors}} \cdot \underbrace{2 \cdot 2}_{\text{2 factors}}$	2^6	4, 2	6
$b^3 \cdot b^5$	$\underbrace{b \cdot b \cdot b}_{\text{3 factors}} \cdot \underbrace{b \cdot b \cdot b \cdot b \cdot b}_{\text{5 factors}}$	b^8	3, 5	8
$5^3 \cdot 5^4$	$\underbrace{5 \cdot 5 \cdot 5}_{\text{3 factors}} \cdot \underbrace{5 \cdot 5 \cdot 5 \cdot 5}_{\text{4 factors}}$	5^7	3, 4	7
$m \cdot m^3$	$\underbrace{m}_{\text{1 factor}} \cdot \underbrace{m \cdot m \cdot m}_{\text{3 factors}}$	m^4	1, 3	4
$a^x \cdot a^y$	$\underbrace{a \cdot a \; \ldots \cdot a}_{\text{x factors}} \cdot \underbrace{a \cdot a \; \ldots \cdot a}_{\text{y factors}}$		x, y	

Notice that the exponent of the simpler numeral has not been given in the last example. After looking at the last two columns for the other examples, do you see that the simpler numeral should be a^{x+y} ?

For all real numbers a, and
for all positive integers x *and* y,

$$a^x \cdot a^y = a^{x+y} .$$

The numeral "a^{x+y}" is considered to be in simpler form than "$a^x \cdot a^y$".

Check Your Reading

1. What does "x^n" mean?
2. What is a simpler numeral for $a^2 \cdot a^3$? for $b^3 \cdot b^5$? for $2^4 \cdot 2^2$?
3. What is a simpler way of writing $a^x \cdot a^y$?

Oral Exercises 11-4a

1. Give simpler names for each of the following numbers.

(a) $2^5 \cdot 2^2$
(b) $3^2 \cdot 3^3$
(c) $a^2 \cdot a^8$
(d) $10^3 \cdot 10^4$
(e) $4^2 \cdot 4$
(f) $a^5 \cdot a^3$
(g) $m \cdot m^4$
(h) 6×6^3
(i) $x^2(x)$
(j) $(y^3)(y^6)$
(k) $x^a \cdot x^b$
(l) $2^m \cdot 2^n$
(m) $(a^x)(a^y)$
(n) $x^2 \cdot x^{3a}$
(o) $(2^2a^2b)(2ab^2)$

2. Which of the following sentences are true? Which are false?

(a) $2^3 + 3^3 = 5^3$
(b) $(2^3)(3^3) = 5^3$
(c) $(2^3)(3^3) = 6^3$
(d) $2^3 + 3^3 = 6^3$
(e) $2^3 = 6$
(f) $2^3 + 2^3 = 2^6$
(g) $2^3 + 2^3 = 4^3$
(h) $2^3 \cdot 2^3 = 4^3$
(i) $2^3 \cdot 2^3 = 2^6$
(j) $3^3 = 9$
(k) $2^3 \times 3^3 = 6 \times 9$
(l) $2^3 + 2^3 = 8 + 8$

3. Which of the following sentences are true for all values of the variables?

(a) $(x^2)^3 = x^5$
(b) $ax^2 = (ax)(ax)$
(c) $(a^2x^3)^3 = a^6x^9$
(d) $(m^2n)^3(m^2n)^4 = (m^2n)^7$

Problem Set 11-4a

1. Write simpler names for each of the following numbers.
   - (a) $2^3 \cdot 2^5$
   - (b) $m^3 \cdot m^6$
   - (c) $3^2 \cdot 3^4$
   - (d) $(a^2)(a^4)$
   - (e) $(m^3)(m^{11})$
   - (f) $x^3 \cdot x^9$
   - (g) $10^5 \times 10^3$
   - (h) $9^2 \times 9$
   - (i) $x(x^3)$
   - (j) $a^{11} \cdot a^{30}$

2. Write simpler names for the following expressions.
   - (a) $2x(2^3x^2)$
   - (b) $(3^3a)(3^4a^3)$
   - (c) $(27a)(81a^3)$
   - (d) $(x^{2a})(x^a)$
   - (e) $3^4 \cdot 3^2$
   - (f) $3^4 \cdot 2^3$
   - (g) $(3a^2b^3)(3^2ab^2)$
   - (h) $(3k^2t)(3m^2t)$

3. Write each of the following expressions as a product of powers of primes and powers of the variables present.

   Example: $27a^2 \cdot 8a^3 = 3^3 \cdot a^2 \cdot 2^3 \cdot a^3$

   $= 3^3 \cdot 2^3 \cdot a^2 \cdot a^3$

   $= 3^3 \cdot 2^3 \cdot a^5$

   - (a) $(16a^2)(32a^4)$
   - (b) $49b(243ab)$
   - (c) $81xy \cdot 3x^2y^3$
   - (d) $17a(34b^2)$
   - (e) $(18m^3n)(24mnp)$
   - (f) $36ab^2(32a^5bc)$
   - (g) $(27x^2)(27x^2)$
   - (h) $(3ab)(3ab)(3ab)$
   - (i) $(2ab)^2$

The definition of exponent also leads to a simpler form for a fraction such as

$$\frac{a^5}{a^3}, \quad \text{where } a \neq 0.$$

Example 1. $\frac{a^5}{a^3} = \frac{a\cdot a\cdot a\cdot a\cdot a}{a\cdot a\cdot a}, \quad a \neq 0$

$$= \left(\frac{a\cdot a\cdot a}{a\cdot a\cdot a}\right)\cdot a\cdot a$$

$$= (1)\cdot a\cdot a$$

$$= a^2.$$

Example 2. $\frac{4x^3y^4}{2xy^2} = \frac{2\cdot 2\cdot x\cdot x\cdot x\cdot y\cdot y\cdot y\cdot y}{2\cdot x\cdot y\cdot y}, \qquad x \neq 0; \quad y \neq 0$

$$= \left(\frac{2\cdot x\cdot y\cdot y}{2\cdot x\cdot y\cdot y}\right)\, 2\cdot x\cdot x\cdot y\cdot y$$

$$= (1)\cdot 2\cdot x\cdot x\cdot y\cdot y$$

$$= 2x^2y^2.$$

In the case of each variable and each number in examples 1 and 2, the exponent in the numerator was greater than the exponent in the denominator. Examples 3 and 4 illustrate what happens when the exponent in the denominator is greater than the exponent in the numerator.

Example 3. $\frac{a^3}{a^5} = \frac{1\cdot a\cdot a\cdot a}{a\cdot a\cdot a\cdot a\cdot a}, \quad a \neq 0$

$$= \left(\frac{a\cdot a\cdot a}{a\cdot a\cdot a}\right)\cdot \frac{1}{a\cdot a}$$

$$= (1)\cdot \frac{1}{a\cdot a}$$

$$= \frac{1}{a^2}.$$

Example 4. $$\frac{9m^2n^2}{54m^4n^5} = \frac{1\cdot3\cdot3\cdot m\cdot m\cdot n\cdot n}{1\cdot2\cdot3\cdot3\cdot3\cdot m\cdot m\cdot m\cdot m\cdot n\cdot n\cdot n\cdot n\cdot n} \qquad m \neq 0;\quad n \neq 0$$

$$= \left(\frac{3\cdot3\cdot m\cdot m\cdot n\cdot n}{3\cdot3\cdot m\cdot m\cdot n\cdot n}\right)\cdot\frac{1}{1\cdot2\cdot3\cdot m\cdot m\cdot n\cdot n\cdot n}$$

$$= (1)\cdot\frac{1}{1\cdot2\cdot3\cdot m\cdot m\cdot n\cdot n\cdot n}$$

$$= \frac{1}{6m^2n^3}$$

The next example illustrates a case in which, for some factors, the greater exponent is in the numerator, and, for others, the greater exponent is in the denominator.

Example 5. $$\frac{6x^3y^2}{9x^2y^5} = \frac{1\cdot2\cdot3\cdot x\cdot x\cdot x\cdot y\cdot y}{1\cdot3\cdot3\cdot x\cdot x\cdot y\cdot y\cdot y\cdot y\cdot y}, \qquad x \neq 0;\quad y \neq 0$$

$$= \left(\frac{1\cdot3\cdot x\cdot x\cdot y\cdot y}{1\cdot3\cdot x\cdot x\cdot y\cdot y}\right)\cdot\frac{2\cdot x}{3\cdot y\cdot y\cdot y}$$

$$= \frac{2x}{3y^3}$$

Examples 1 and 3 illustrate the basic ideas used in all of the other examples.

If $a \neq 0$, $$\frac{a^5}{a^3} = a^{5-3}$$ The greater exponent is in the numerator.

$$= a^2$$

If $a \neq 0$, $$\frac{a^3}{a^5} = \frac{1}{a^{5-3}}$$ The greater exponent is in the denominator.

$$= \frac{1}{a^2}$$

Turning from the specified exponents 3 and 5 to the variables m and n, we are able to say that the following simplifications may be made:

For any real number a, $a \neq 0$,
and for any positive integers m and n,

$$\frac{a^m}{a^n} = a^{m-n} \qquad \text{if } m > n,$$

$$\frac{a^m}{a^n} = 1 \qquad \text{if } m = n,$$

$$\frac{a^m}{a^n} = \frac{1}{a^{n-m}} \qquad \text{if } n > m.$$

Check Your Reading

1. What is a simpler name for $\frac{a^5}{a^3}$?
2. What is a simpler name for $\frac{a \cdot a \cdot a}{a \cdot a \cdot a}$?
3. What is a simpler name for $\frac{a^3}{a^5}$?
4. For what values of x and y is the following true?

$$\frac{6x^3y^2}{9x^2y^5} = \frac{2x}{3y^3}$$

5. What is a simpler name for $\frac{a^m}{a^n}$, $a \neq 0$, if
   (a) $m = n$?
   (b) $n > m$?
   (c) $m > n$?

Oral Exercises 11-4b

1. Give a simpler form for each of the following expressions, as was done in the examples of this section. (None of the variables in a denominator can have the value 0.)

(a) $\frac{m^5}{m^2}$
(b) $\frac{3x^2}{2x}$
(c) $\frac{a^{11}}{a^2}$
(d) $\frac{6ax^7}{2x^3}$
(e) $\frac{a^2}{a^{11}}$
(f) $\frac{ay^2}{y^4}$
(g) $\frac{3a^2m}{4am^2}$
(h) $\frac{2^3a^2b}{2ab}$
(i) $\frac{3^4y^4}{3y^3}$
(j) $\frac{2^25^2m}{2^35am^3}$
(k) $\frac{42x^3y}{14xy^3}$
(l) $\frac{2^2\cdot 3^2\cdot 5\cdot 7^4}{2^3\cdot 3^3\cdot 5^3\cdot 7}$
(m) $\frac{100m^2x}{(32)(5x^2)}$

2. Which of the following sentences are true? Which are false?

(a) $\frac{3^2}{2^2} = \frac{3}{2}$
(b) $\frac{6^3}{3^3} = 2$
(c) $\frac{2^4\cdot 3}{2\cdot 3^2} = \frac{2^3}{3}$
(d) $(\frac{4^2}{3^2})(\frac{3}{4}) = 1$
(e) $\frac{6^3}{3^3} = 2^3$

3. Which of the following sentences are true for all values of the variables?

(a) $5x^2 = (25)(x^2)$
(b) $(2^2m^3)(2m^3) = 2^3m^3$
(c) $(2m^2)(2^3m^3) = 2^4m^5$
(d) $\frac{2^3m^3}{2m^2} = 2^2m$

4. Why must we not allow any of the variables appearing in a denominator to have the value 0?

Problem Set 11-4b

Simplify each of the following expressions. (None of the variables appearing in a denominator can have the value 0.)

1. $\frac{m^{11}}{m^3}$

2. $\frac{2x^4}{2^3x^2}$

3. $\frac{(5^3x^2)(5x)}{5^2x^6}$

4. $\frac{a^6b^6}{ab^2}$

5. $a^7b^3c^2 \cdot abc^3$

6. $\frac{36a^2b^3}{8a^5b}$

7. $\frac{288x^2y^3}{48x^6y^6}$

8. $\frac{150h^2m^8}{225h^2t^8}$

9. $\frac{81ax^2}{16a^4x^9}$

10. $\frac{22a^3b^3c^3}{16a^3bcy^2}$

11. $\frac{24h^2c^3y}{16h^3bcy^2}$

12. $\frac{49a^4bc^3}{14a^2b^2c^2}$

13. $\frac{50c^4d^3y^2}{-5c^3dy^4}$

14. $\frac{36x^2y^4z^2}{72x^3yz}$

15. $\frac{36a^2b}{24ab^2}$

16. $\frac{256a}{288a^2b}$

17. $\frac{(-2)^3a^2b^7}{8a^2b^5}$

18. $\frac{8 \cdot 2^3m^2n}{97mn^2}$

19. $\frac{-2^2a^2m}{128a^2m^3}$

20. $\frac{(-2)^2a^2m}{128a^2m^3}$

21. $\frac{x^{2a}}{x^a}$

22. $\left(\frac{3^2}{4}\right)\left(\frac{28a^3}{45a}\right)$

Problem Set 11-4b
(continued)

23. $\left(\frac{m^3n^2}{m}\right)\left(\frac{n^3}{m}\right)$

24. $\left(\frac{2^3 \cdot 3^4}{5^2}\right)\left(\frac{5^3}{2^6 \cdot 3^2}\right)$

*25. $\left(\frac{63a^2}{243a^5}\right)\left(\frac{54a^7}{14a^4}\right)$

*26. $\dfrac{\frac{90(ab)^2}{16a^3}}{\frac{108ab^3}{81}}$

*27. $\dfrac{\frac{-30xy}{85x^2}}{\frac{150(-y)}{34z}}$

*28. $\dfrac{\frac{256a^8}{2^8a^8}}{\frac{25ab}{5(5a + 10b)}}$

29. Which of the following statements are true? Which are false? Give a reason for your answer.

(a) $(\frac{2}{3})^2 = \frac{2^2}{3^2}$

(b) $\frac{2}{3} = \frac{2^2}{3^2}$

(c) 3^3 is a factor of $(3^3 + 5^3)$

(d) 3^2 is a factor of $(6^2 + 9^2)$

(e) $(2x + 4y^2)$ is an even number if x and y are positive integers.

The expression

$$(ab)^3$$

is easily written in another form by using the definition of an exponent together with the commutative and associative properties of multiplication.

$$\begin{aligned}(ab)^3 &= (ab)(ab)(ab)\\ &= (aaa)(bbb)\\ &= a^3b^3.\end{aligned}$$

In the same way, it can be shown that $(ab)^2 = a^2b^2$ and $(ab)^4 = a^4b^4$. In fact, for any real numbers a and b, and for any positive integer n, it is true that

$$(ab)^n = a^nb^n.$$

Do you see that this conclusion is a result of previously accepted properties and definitions? A somewhat more complicated example is given below.

Example 1.

$$\begin{aligned}(a^2b^3)^3 &= (a^2)^3(b^3)^3\\ &= (a^2)(a^2)(a^2)(b^3)(b^3)(b^3)\\ &= a^6b^9.\end{aligned}$$

Closely related to the above discussion is the problem of simplifying an expression such as

$$(\frac{a}{b})^3, \quad b \neq 0.$$

$$\begin{aligned}(\frac{a}{b})^3 &= (\frac{a}{b})(\frac{a}{b})(\frac{a}{b})\\ &= \frac{a^3}{b^3}.\end{aligned}$$

Similarly, $(\frac{a}{b})^2 = \frac{a^2}{b^2}$, $(\frac{a}{b})^4 = \frac{a^4}{b^4}$, and for any real numbers a and b, and for any positive integer n,

$$(\frac{a}{b})^n = \frac{a^n}{b^n}, \qquad b \neq 0.$$

Example 2. Simplify $(\frac{2a^2}{b^3})^2$, where $b \neq 0$.

$$\begin{aligned}(\frac{2a^2}{b^3})^2 &= (\frac{2a^2}{b^3})(\frac{2a^2}{b^3})\\ &= \frac{4a^4}{b^6}.\end{aligned}$$

Problem Set 11-4c

Write simpler names for these numbers. (None of the variables appearing in the denominator can have the value 0.)

1. $(a^2)^5$
2. $(d^3)^3$
3. $(ab^2)^4$
4. $(x^2y)^3$
5. $(2am^3)^2$
6. $(3c^3d^2)^3$
7. $(5a)^2(a^2)$
8. $(3x^2)^3(x^2)^2$
9. $\frac{(r^2s^4)^3}{(rs^2)^2}$
10. $\frac{(a^2)^5}{(a^2)^2}$
11. $\frac{18a^2m}{3(am)^2}$
12. $\frac{(5x)^3}{3x(25x)^2}$
13. $\frac{(3a)^2}{12ab^2}$
14. $\frac{(-3a)^2}{9}$
15. $\frac{-3a^2}{9}$
16. $\frac{(-2x^2yz)^3}{(-2x^2yz)^2}$
17. $\frac{(-5)^2(a^2c)^3}{(5ac)^2}$
18. $\frac{(-a^3k)^2}{a^4(-k)^3}$
19. $\left(\frac{ab}{c}\right)^2\left(\frac{c^2}{a^3b}\right)$
20. $\left(\frac{2}{3}\right)^2\left(\frac{3b}{2}\right)^3$

Find the value of each of the following expressions if a is 2, b is -2, c is 3, and d is -3. First write the expression in a simpler form if possible.

21. $-2a^2b^2c^2$
22. $(-2abc)^2$
23. $\frac{-4a^4d}{6a^3b^2}$
24. $\frac{a^2b^2c^4}{4a^3bc^2}$

Problem Set 11-4c
(continued)

25. $\dfrac{(ab)^3}{(a^2b^2)^3}$

*26. $\dfrac{(a + b)^3}{a^3 + b^3}$

*27. $\dfrac{-6a^{12}b^{16}c^{20}}{2a^{10}b^{18}c^{22}}$

*28. $\dfrac{(a + b + c)^2}{a^2 + b^2 + c^2}$

11-5. Adding and Subtracting Fractions.

It is easy to add or to subtract two fractions if their denominators are the same number. For example,

$$\begin{aligned}\tfrac{3}{8} + \tfrac{2}{8} &= 3(\tfrac{1}{8}) + 2(\tfrac{1}{8})\\ &= (3 + 2)(\tfrac{1}{8})\\ &= (5)(\tfrac{1}{8})\\ &= \tfrac{5}{8}\ .\end{aligned}$$

Furthermore, when the multiplication property of one is used, adding or subtracting fractions whose denominators are not the same number is also simple. For example, consider the sum

$\frac{3}{4} + \frac{5}{8}$. This is the indicated sum of two fractions that do not have the same denominator.

But $\frac{3}{4} = (\frac{2}{2})(\frac{3}{4})$ The multiplication property of one has been used here, with the numeral $\frac{2}{2}$ for one.

$= \frac{6}{8}$.

$$\text{So}\quad \frac{3}{4} + \frac{5}{8} = \frac{6}{8} + \frac{5}{8}$$

$$= \frac{11}{8}.$$

In the second example above, the problem was restated so that both fractions in the indicated sum had the number 8 as a denominator. Many other integers could have been chosen for the denominator--for example, 16, 24, or 32--but 8 is the smallest integer that could have been chosen. The number 8 is the least common multiple of the numbers 4 and 8. The idea of least common multiple arises at this time because it is based on factoring. For example, what is the least common multiple of 2 and 3?

Positive multiples of 2: {2, 4, 6, 8, 10, 12, 14, ...}.
Positive multiples of 3: {3, 6, 9, 12, 15, 18, 21, ...}.

Least common multiple of 2 and 3: 6.

Example 2. Find the least common multiple of 18 and 9.

Positive multiples of 18: {18, 36, 54, 72, 90, ...}.
Positive multiples of 9: {9, 18, 27, 36, 45, 54, ...}.

Least common multiple of 18 and 9: 18.

Example 3. Find the least common multiple of 8 and 12.

Positive multiples of 8: {8, 16, 24, 32, 40, 48, ...}.
Positive multiples of 12: {12, 24, 36, 48, 60, 72, ...}.

Least common multiple of 8 and 12: 24.

From the examples above, it can be seen that the least common multiple of two numbers is the smallest positive integer that is a multiple of both of the numbers. Because of the relationship between "multiple" and "factor", the least common multiple may also be defined in this way:

> The least common multiple of two integers is the smallest positive integer that has each of them as factors.

Check Your Reading

1. List the set of common multiples of 2 and 3.
2. What is the least common multiple of 2 and 3?
3. What is the least common multiple of 8 and 12?
4. Is it possible for the least common multiple of two numbers to be one of the given numbers? Give an illustration.
5. Is it possible for the least common multiple of two numbers to be the product of the two numbers? Give an illustration.

Oral Exercises 11-5a

For each set of numbers give the first 5 positive multiples of each number of the set. Then tell whether you have named the least common multiple of the numbers of the set.

1. 4, 5
2. 2, 10
3. 9, 12
4. 3, 5, 6
5. 2, 7, 12
6. 2, 6, 11
7. 5, 10, 15
8. a, $3a$
9. $2a$, $3a$
10. x^2, $2x^2$

Problem Set 11-5a

Using the method illustrated in this section, find the least common multiple of the numbers in each set.

1. 3, 5
2. 4, 6, 8
3. 12, 15
4. 10, 15, 25
5. 2, 10, 25
6. 4, 18, 20
7. 30, 36
8. $2a$, $8a$
9. $2x$, $5x$, $6x$
10. $2x$, $3y$

It is possible to develop a more systematic method for determining the least common multiple. To see why a more systematic method might be desirable, consider the problem of finding the least common multiple of the numbers 6, 8, and 27.

Positive multiples of 6: 6, 12, 18, 24, 30, 36, 42, 48, 54, ...
Positive multiples of 8: 8, 16, 24, 32, 40, 48, 56, 64, 72, ...
Positive multiples of 27: 27, 54, 81, 108, 135, 162, ...

So far, the listing of multiples has not uncovered a common multiple at all. In time, one would surely turn up (the product of the three numbers is certainly a common multiple), but you can see that this might be a time-consuming process. A quicker method is detailed below.

$$6 = 2\cdot 3$$
$$8 = 2\cdot 2\cdot 2 = 2^3$$
$$27 = 3\cdot 3\cdot 3 = 3^3$$

The prime factorization of each of the three numbers has been written.

Suppose we let n represent the least common multiple of the three numbers. Since n is a multiple of 6, of 8, and of 27, then by the relationship between multiple and factor, 6, 8, and 27 are factors of n. Therefore, the prime factorization of n can be determined from the prime factorizations above.

Since 6 is a factor of n, 2 and 3 must be factors of n. The prime factorization of n may be started (not completed) like this:

$\underbrace{2\cdot 3}$ _______ These factors make n a multiple of 6.

Since 8 is a factor of n, $2\cdot 2\cdot 2$ must be a factor of n. One factor 2 has already been introduced, two more are needed, giving

$\underbrace{2\cdot 3\cdot 2\cdot 2}$ _______ These factors make n a multiple of 8.

Since 27 is a factor of n, $3\cdot 3\cdot 3$ must be a factor of n. One factor 3 has already been introduced. Two more are needed, giving

$2\cdot\underbrace{3\cdot 2\cdot 2\cdot 3\cdot 3}$ These factors make n a multiple of 27.

$2\cdot 3\cdot 2\cdot 2\cdot 3\cdot 3$ then is a common multiple of 6, 8, and 27. It is the least common multiple because removing any one of the factors would mean that n would no longer be a multiple of each of the three numbers.

Rearranging the factors and introducing exponents, the least common multiple of 6, 8, and 27 may now be written

$$2^3 \cdot 3^3 .$$

Notice that in the prime factorizations of 6, 8, and 27 the only factors were 2 and 3. The greatest power of 2 was 2^3 (occurring in the factorization of 8), and the greatest power of 3 was 3^3 (occurring in the factorization of 27). The least common multiple is the product of these "greatest powers".

As another example of this systematic method of determining the least common multiple, consider the problem of finding the least common multiple of the numbers 12, 15, 16, and 25.

$$12 = 2\cdot 2\cdot 3 = 2^2 \cdot 3$$

$$15 = 3\cdot 5$$

$$16 = 2\cdot 2\cdot 2\cdot 2 = 2^4$$

$$25 = 5\cdot 5 = 5^2$$

Notice that the only factors that occur are 2, 3, and 5, and

the highest power of 2 is 2^4,

the highest power of 3 is 3^1 (first power),

the highest power of 5 is 5^2.

The least common multiple of 12, 15, 16, and 25 is

$$2^4 \cdot 3 \cdot 5^2 .$$

A common name for this least common multiple is 1200. However, in much of our future work, it will be convenient to leave it in the factored form above.

Oral Exercises 11-5b

State the least common multiples of these sets of numbers in factored form.

1. 2×3, 2×5, 3×5
2. $2 \times 2 \times 3$, $3 \times 3 \times 5$, $5 \times 5 \times 7$
3. 2×7, $2 \times 3 \times 3$, $2 \times 2 \times 7$, 7×7
4. 3×11, $3 \times 3 \times 5$, $2 \times 2 \times 13$, $3 \times 11 \times 13$
5. 3, 4
6. 3, 4, 6
7. 3, 4, 6, 9
8. 5, 25, 15
9. 14, 49, 4
10. 10, 25, $3 \times 5 \times 5$, 49
11. $2^2 \cdot 3$, $3^2 \cdot 5$, $2^3 \cdot 3 \cdot 5^2$

Problem Set 11-5b

Write the prime factorization of the least common multiple of the set of numbers in each of the following.

1. 25, 10, 4
2. 12, 8, 16
3. 9, 14, 21
4. 8, 12, 15
5. 10, 16, 20
6. 6, 10, 20, 25
7. 9, 16, 24, 27
8. 8, 34, 22, 51
9. 65, 4, 52, 26
10. 3, $2x$, $3x^2$

With "least common multiple" defined, we can return to the problem of adding and subtracting fractions. We can now place the solution of such problems on a much firmer mathematical foundation.

Example 1. What is a common name for $\frac{3}{10} - \frac{4}{45} + \frac{1}{6}$?

The multiplication property of one can be applied to find a new name for each of the three fractions in the indicated sum. The question of what denominator these new numerals shall have is answered by finding the least common multiple of the denominators 10, 45, and 6. This is often called the least common denominator of the fractions.

$$10 = 2 \cdot 5$$

$$45 = 3 \cdot 3 \cdot 5 = 3^2 \cdot 5$$

$$6 = 2 \cdot 3$$

The least common multiple is $2 \cdot 3^2 \cdot 5$. This means that we want to find a name with denominator $2 \cdot 3 \cdot 3 \cdot 5$ for each of the fractions in the indicated sum.

$$\frac{3}{10} = \frac{3}{2 \cdot 5}$$

Instead of the denominator $2 \cdot 5$, the denominator $2 \cdot 3 \cdot 3 \cdot 5$ is called for. This means we must multiply by the factor $3 \cdot 3$. By the multiplication property of one, then, the fraction is multiplied by $(\frac{3 \cdot 3}{3 \cdot 3})$.

$$\begin{aligned} \frac{3}{10} &= \frac{3}{2 \cdot 5} \\ &= \left(\frac{3}{2 \cdot 5}\right) \cdot \left(\frac{3 \cdot 3}{3 \cdot 3}\right) \\ &= \frac{3 \cdot 3 \cdot 3}{2 \cdot 3 \cdot 3 \cdot 5} \end{aligned}$$

This is the desired numeral for $\frac{3}{10}$.

Similarly, new numerals for $\frac{4}{45}$ and $\frac{1}{6}$ are found below.

$$\begin{aligned} \frac{4}{45} &= \frac{4}{3 \cdot 3 \cdot 5} \\ &= \left(\frac{4}{3 \cdot 3 \cdot 5}\right) \cdot \left(\frac{2}{2}\right) \\ &= \frac{4 \cdot 2}{2 \cdot 3 \cdot 3 \cdot 5} \end{aligned}$$

$$\begin{aligned} \frac{1}{6} &= \frac{1}{2 \cdot 3} \\ &= \left(\frac{1}{2 \cdot 3}\right) \cdot \left(\frac{3 \cdot 5}{3 \cdot 5}\right) \\ &= \frac{3 \cdot 5}{2 \cdot 3 \cdot 3 \cdot 5} \end{aligned}$$

Now the original problem can be restated and a simpler numeral found.

$$\begin{aligned}\frac{3}{10} - \frac{4}{45} + \frac{1}{6} &= \frac{27}{2\cdot3\cdot3\cdot5} - \frac{8}{2\cdot3\cdot3\cdot5} + \frac{15}{2\cdot3\cdot3\cdot5} \\ &= \frac{34}{2\cdot3\cdot3\cdot5} \\ &= \frac{2\cdot17}{2\cdot3\cdot3\cdot5} \\ &= \frac{17}{3\cdot3\cdot5} \\ &= \frac{17}{45}\end{aligned}$$

In this example, the least common denominator was 90. Notice, however, that it was left in factored form until the last step.

Example 2. Find a common name for $\frac{1}{3} + \frac{5}{12} - \frac{11}{20}$.

The denominator 3 is a prime number. The prime factorizations of the other two denominators are:

$2\cdot2\cdot3$, $2\cdot2\cdot5$

Least common denominator: $2\cdot2\cdot3\cdot5$

$$\begin{aligned}\frac{1}{3} + \frac{5}{12} - \frac{11}{20} &= \frac{1}{3}\left(\frac{2\cdot2\cdot5}{2\cdot2\cdot5}\right) + \frac{5}{2\cdot2\cdot3}\left(\frac{5}{5}\right) - \frac{11}{2\cdot2\cdot5}\left(\frac{3}{3}\right) \\ &= \frac{20}{2\cdot2\cdot3\cdot5} + \frac{25}{2\cdot2\cdot3\cdot5} - \frac{33}{2\cdot2\cdot3\cdot5} . \\ &= \frac{12}{2\cdot2\cdot3\cdot5} \\ &= \frac{1}{5}\end{aligned}$$

Check Your Reading

1. What is the relationship between "least common multiple" and "least common denominator"?

Check Your Reading
(continued)

2. If we have the fraction $\frac{3}{2 \cdot 5}$, what numeral for "1" do we use in order to write this as a fraction with the denominator $2 \cdot 3 \cdot 3 \cdot 5$?

3. If we have the fraction $\frac{1}{2 \cdot 3}$, what numeral for "1" do we use to write it in the form $\frac{1 \cdot 3 \cdot 5}{2 \cdot 3 \cdot 3 \cdot 5}$?

4. What name for the fraction $\frac{11}{20}$ has $2^2 \cdot 3 \cdot 5$ as denominator?

Oral Exercises 11-5c

1. What is the least common multiple of the numbers in each of the following sets of numbers?

   (a) 6, 12, 18
   (b) 25, 5, 10
   (c) 26, 13, 4
   (d) 2, 44, 8
   (e) 20, 30, 40

2. What is the least common denominator of each of the following sets of fractions?

   (a) $\frac{1}{5}, \frac{2}{3}, \frac{4}{15}, \frac{1}{2}$
   (b) $\frac{5}{12}, \frac{3}{11}, \frac{3}{22}$
   (c) $\frac{1}{14}, \frac{1}{28}, \frac{1}{4}$
   (d) $\frac{3}{4}, \frac{5}{8}, \frac{2}{3}, \frac{1}{6}$
   (e) $\frac{1}{20}, \frac{1}{30}, \frac{1}{40}$

3. Tell what form of "1" we will have to multiply by to produce fractions having the indicated denominators. Then tell what the numerators will be.

   (a) $\frac{3}{2 \times 3}(\underline{\quad\quad}) = \frac{\quad}{2 \times 2 \times 3 \times 3 \times 5}$
   (b) $\frac{1}{2 \times 2 \times 7}(\underline{\quad\quad}) = \frac{\quad}{2 \times 2 \times 7 \times 11}$
   (c) $\frac{15}{5 \times 7 \times 7}(\underline{\quad\quad}) = \frac{\quad}{2 \times 2 \times 5 \times 5 \times 7 \times 7}$

Oral Exercises 11-5c
(continued)

(d) $\frac{1}{12}(\text{——}) = \frac{\quad}{2 \times 2 \times 3 \times 5 \times 7}$

(e) $\frac{3}{15}(\text{——}) = \frac{\quad}{2 \times 3 \times 3 \times 5 \times 5}$

(f) $\frac{36a}{25}(\text{——}) = \frac{\quad}{2 \times 2 \times 5 \times 5}$

Problem Set 11-5c

Find a common name for each of the following:

1. $\frac{5}{6} + \frac{7}{9}$
2. $\frac{1}{32} + \frac{5}{40}$
3. $\frac{7}{36} - \frac{4}{15} + \frac{1}{2}$
4. $\frac{11}{26} - \frac{5}{65} + \frac{7}{10}$
5. $\frac{1}{85} + \frac{3}{51}$
6. $\frac{5}{9} - \frac{26}{75}$
7. $\frac{1}{6} + \frac{3}{20} - \frac{2}{45}$

*8. $\frac{-20}{57} - \frac{7}{38}$

*9. $\frac{5}{21} - \frac{3}{91}$

*10. $\frac{5}{34} + \frac{11}{85} - \frac{7}{10}$

You may have noticed that much of the work in finding least common denominators can be done without paper and pencil. For example, consider the problem of finding a common name for

$$\frac{1}{5} + \frac{7}{15} + \frac{2}{25} + \frac{5}{9}.$$

The factor 3 occurs two times in 9, the factor 5 two times in 25. There are no prime factors other than 3 and 5. So the lowest common denominator is $3 \cdot 3 \cdot 5 \cdot 5$.

The first fraction "needs" the factors, 3, 3, and 5 for the least common denominator.
The second fraction "needs" the factors 3 and 5.
The third fraction "needs" the factors 3 and 3.
The fourth fraction "needs" the factors 5 and 5.

The following is all that needs to be written:

$$\begin{aligned}\frac{1}{5} + \frac{7}{15} + \frac{2}{25} + \frac{5}{9} &= \frac{1(3\cdot 3\cdot 5) + 7(3\cdot 5) + 2(3\cdot 3) + 5(5\cdot 5)}{3\cdot 3\cdot 5\cdot 5}\\ &= \frac{45 + 105 + 18 + 125}{3\cdot 3\cdot 5\cdot 5}\\ &= \frac{293}{225}.\end{aligned}$$

Example. For any real number x,

$$\begin{aligned}\frac{3x}{10} + \frac{x}{12} + \frac{3}{8} &= \frac{3x(2\cdot 2\cdot 3) + x(2\cdot 5) + 3(3\cdot 5)}{2\cdot 2\cdot 2\cdot 3\cdot 5}\\ &= \frac{36x + 10x + 45}{2\cdot 2\cdot 2\cdot 3\cdot 5}\\ &= \frac{46x + 45}{120}.\end{aligned}$$

Problem Set 11-5d

1. Express each of the following sums as a single fraction in simplest form.

(a) $\frac{x}{20} + \frac{5}{24}$

(b) $\frac{3x}{8} + \frac{5x}{36}$

(c) $\frac{a}{27} - \frac{7a}{24}$

(d) $\frac{3m}{25} + \frac{m}{10} + \frac{3}{4}$

(e) $\frac{2c}{5} - \frac{11c}{75} + \frac{4c}{9}$

(f) $n^2 + \frac{n}{28} - \frac{3}{98}$

(g) $\frac{1}{4} + \frac{2d}{10} + \frac{5d}{14}$

(h) $\frac{3k}{10} + \frac{2k}{28} - \frac{k}{56}$

(i) $\frac{x}{3} + \frac{5x}{8} - \frac{7x}{18}$

*(j) $\frac{x}{22} - \frac{a}{12} + a$

*(k) $\frac{3a}{5} + \frac{7a}{75} - \frac{5a}{63}$

*(l) $\frac{x}{4} + \frac{5x}{8} - \frac{11}{70} + \frac{3}{20}$

2. Determine which of the following sentences are true and which are false.

(a) $\frac{8}{15} < \frac{13}{24}$

(b) $\frac{3}{16} < \frac{11}{64}$

(c) $\frac{14}{63} < \frac{6}{27}$

(d) $\frac{35}{60} = \frac{49}{84}$

(e) $\frac{26}{32} < \frac{28}{33}$

Problem Set 11-5d
(continued)

3. In each of the following, determine which number is the greater.

(a) $\frac{1}{7}$ or $\frac{1}{2} - \frac{1}{3}$

(b) $\frac{4}{15}$ or $\frac{7}{27}$

(c) $\frac{5}{12}$ or $\frac{5}{13}$

(d) $\frac{1}{67}$ or $\frac{2}{134}$

(e) $-\frac{3}{24}$ or $-\frac{5}{40}$

(f) $-\frac{5}{8}$ or $\frac{10}{16}$

(g) .49 or .499

(h) - .009 or - .0009

(i) .900 or .09000

(j) .49 or $\frac{15}{30}$

4. You have learned in Chapter 9 that for any pair of numbers a and b, exactly one of the following must hold: $a > b$, $a = b$, or $a < b$. Put in the symbol for the correct relation for the pairs of numbers below:

(a) $\frac{6}{27}$, $\frac{5}{28}$

(b) $\frac{2}{3}$, $\frac{5}{7}$

(c) $\frac{6}{16}$, $\frac{9}{24}$

(d) $(\frac{1}{2} + \frac{1}{3})$, $(\frac{11}{12} - \frac{1}{13})$

5. A man is hired to sell suits at the AB Clothing Store. He is given the choice of being paid $200 plus $\frac{1}{12}$ of his sales or a straight $\frac{1}{3}$ of his sales. If he thinks he can sell $600 worth of suits per month, which is the better choice? Suppose he could sell $700 worth of suits, which is the better choice? What if he could sell $1000 worth? What should his sales be so that the offers are equal?

*6. John and his brother Bob each received an allowance of $1.00 per week. One week their father said, "I will pay each of you $1.00 as usual or I will pay you in cents any number less than 100 plus its largest prime factor which is also a proper factor. If you are not too lazy, you have much to gain." What number should they choose?

Summary

1. A multiple of a number is the product of that number and an element from the set of integers.
2. A positive integer x is a factor of a positive integer y if y is a multiple of x.
3. A positive integer y is divisible by a positive integer x if x is a factor of y.
4. In the set of positive integers, any number has itself and the number one as factors. All other factors are called proper factors.
5. A positive integer that has no proper factors is called a prime number.
6. Every integer greater than one and not prime can be written as an indicated product in which every factor is a prime number. This indicated product is called the prime factorization of the number.
7. In the set of positive integers, the prime factorization of a number is unique.
8. In a phrase of the form "x^n", the number n is called an exponent, and the number x is called the base. An exponent indicates how many times the base is to be used as a factor. x^n itself is called the n^{th} power of x.
9. If $a \neq 0$ and if m and n are positive integers, then the following statements are true:

$$a^m \cdot a^n = a^{m+n}$$

$$\frac{a^m}{a^n} = a^{m-n}, \qquad \text{if } m > n$$

$$\frac{a^m}{a^n} = 1, \qquad \text{if } m = n$$

$$\frac{a^m}{a^n} = \frac{1}{a^{n-m}}, \qquad \text{if } n > m$$

10. The *least common multiple* of a set of positive integers is the smallest positive integer that has each of the numbers as a factor. The *least common denominator* of a set of fractions is simply the least common multiple of the denominators of the fractions.

Review Problem Set

1. Name one proper factor of each of these numbers without doing any dividing.

   (a) 432

   (b) 1884

   (c) 928431

   (d) 920040

   (e) 62511

   (f) 19725

2. Find the prime factorization of each of the following numbers. Use exponents where appropriate.

   (a) 180

   (b) 378

   (c) 3075

   (d) 792

   (e) 6384

3. (a) Is 36 a factor of 414216?

   (b) Is 25 a factor of 732475?

   (c) Is 15 a factor of 43125?

   (d) Is 27 a factor of 81729?

   Hint: These can be done mentally if you think about divisibility principles we have discussed and look for short-cuts.

4. Write each of the following in simplest form.

   (a) $b \cdot b \cdot b \cdot b \cdot b \cdot b$

   (b) $x \cdot x \cdot x \cdot x \cdot m \cdot m \cdot m$

   (c) $\frac{n \cdot n \cdot n \cdot n}{n \cdot n}$

   (d) $\frac{y \cdot y \cdot y}{y \cdot y \cdot y \cdot y}$

   (e) $(3a^2b)(3^3a^5b^2x)$

   (f) $\frac{6a^5x^2}{2ax^4}$

   (g) $\frac{(14m^2n^2)(18m^5n)}{(49m^8n)(36n^3)}$

   (h) $\frac{(x^2 + 4)^2}{(x^2 + 4)}$

   (i) $\frac{(x - y)(x + y)^3}{(x - y)^2(x + y)}$

   (j) $\frac{15a^2x(a + b)^5}{5^2a^6(a + b)^2}$

   (k) $(-2a)^5$

Review Problem Set
(continued)

5. Express each of the following sums in a simpler form.

   (a) $\frac{1}{15} + \frac{3}{8} + \frac{4}{20}$

   (b) $\frac{x}{4} + \frac{3x}{12} - \frac{7x}{18}$

   (c) $\frac{ax}{6} + \frac{bx}{15} + \frac{cx}{14}$

   (d) $\frac{3m}{5} - \frac{7n}{28} - \frac{2c}{35}$

   (e) $\frac{14y}{2} + \frac{x + y}{24} - \frac{x - y}{30}$

6. Which of the following sentences are true? Which are false?

   (a) 15 is a factor of 123345.

   (b) The prime factorization of 36 is 4×9.

   (c) $\frac{8^3}{4^3} = 2^3$

   (d) The least common multiple of $2^3 \cdot 3^2$, $2^5 \cdot 5^2$, and $2 \cdot 3^2 \cdot 5$ is $2^5 \cdot 3^3 \cdot 5^2$.

   (e) To change "$\frac{4}{15}$" to a fraction whose denominator is $2^2 \cdot 3^3 \cdot 5^4$ multiply by $\frac{2^2 \cdot 3^2 \cdot 5^3}{2^2 \cdot 3^2 \cdot 5^3}$

7. Which of the following open sentences are true for all values of the variables?

   (a) $a^5b^4 = (a + a + a + a + a)(b + b + b + b)$, $a \neq 0$, $b \neq 0$

   (b) $(m^2n^3)^4 = m^8n^{12}$

   (c) $\frac{a^2}{a^5} = a^3$, $a \neq 0$

   (d) $3x^3 = 3 \cdot 3 \cdot 3 \cdot x \cdot x \cdot x$, $x \neq 0$

8. Translate these phrases or sentences into algebra. Indicate what the variables represent.

   (a) The sum of two numbers multiplied by the difference of the same two numbers.

   (b) The quotient of two numbers increased by twice the product of the two numbers.

Review Problem Set
(continued)

(c) A number increased by 7 is three times as large as twice the number.

(d) The distance traveled in 2 hours at a certain speed is equal to the distance traveled in 3 hours at a speed 5 miles per hour less than the first.

(e) The total length of a piece of wood and another piece 5" more than twice as long is 76 inches.

(f) 10% of a certain number of pounds of solution is the amount of salt in the solution.

(g) A dining table was placed on sale at a discount of 10% of the regular price, and the reduced price was $95.

(h) A mixture is made from two kinds of candy, one selling for $2 per pound, the other for $1 per pound. The mixture contains a certain number of pounds of the $2 per pound candy. The amount of $1 candy is five pounds more than the amount of $2 candy. The total cost of the mixture is $23.

(i) A man walks in one direction at 2 miles per hour and another, starting from the same place at the same time, walks in the opposite direction at 3 miles per hour. After a certain length of time has elapsed they are 14 miles apart.

9. The sum of three consecutive even integers is 5920. Find the numbers.

10. The perimeter of a rectangle is 47 feet. The width is $7\frac{1}{4}$ feet. Find the length.

11. One side of a triangle is twice as long as another side and the third side is $1\frac{1}{2}$ times as long as the longer of the first two. The perimeter is 87 inches. How long is each side?

Review Problem Set
(continued)

12. Find the truth sets of the following open sentences.

    (a) $3x + 2 - 5x + 7 = 3(x + 4) - 8$

    (b) $2x - 5 > x + 7$

    (c) $3y + 4 - (y - 2) = -4(-2y + 3)$

    (d) $\frac{y}{5} + 2y = \frac{1}{3} + \frac{7y}{2}$

    (e) $x + 4 < 2x + 7$

    (f) $|x| = 4$

    (g) $|x + 1| = -3$

    (h) $|x - 2| = 2$

    (i) $(x + 5)(x - 2) = 0$

    (j) $3x + 5 > 3x + 2$

13. Find two integers

    (a) whose product is 96 and whose sum is 35
    (b) whose product is 96 and whose sum is 28
    (c) whose product is 600 and whose sum is 70
    (d) whose product is 600 and whose sum is 83
    (e) whose product is 600 and whose sum is 601
    (f) whose product is 600 and whose sum is 48

14. Find the dimensions of a rectangle whose area is 800 square feet and whose perimeter is 120 feet.

15. Find the base and the altitude of a triangle if their sum is 21 units and the area of the triangle is 52 square units.

Chapter 12

RADICALS

12-1. Square Roots.

In our study of the addition of one real number to another we found that we can always reverse, or "undo" the process. This can be done by adding the opposite. For example, adding 5 can be reversed by adding (-5).

In multiplication the same thing is true, with one exception. Multiplication by any real number, except zero, can always be reversed. This can be done by multiplying by the reciprocal. For example, multiplication by 5 can be reversed if we multiply by $\frac{1}{5}$.

Let us now consider the operation of squaring a number. You recall that this is the multiplication of a number by itself. What about the reverse of this operation? Suppose we begin with a real number which we assume is the square of some other real number. How do we determine what that other number is?

To put this problem in terms of a mathematical sentence we might say, for example, given the sentence

$$x^2 = 49,$$

find a value of x which will make this a true sentence. It should be clear that one element in the truth set is 7. Can you find another one? We see that it is -7. We can check this by noting that $(7)^2 = 49$, and $(-7)^2 = 49$.

As another example, consider the sentence

$$y^2 = 100.$$

It should be clear that 10 and -10 are elements of the truth set of this sentence. Are these the only elements? To answer this question we note that if x is another positive number besides 10, then either $x < 10$, or $x > 10$. But by a property of order we can say that

if $x < 10$, then $x^2 < 100$, and if $x > 10$, then $x^2 > 100$.

Thus we see that 10 is the only positive element.

In connection with the negative number -10 it can be shown that if x is a negative number such that

$$x < -10, \quad \text{then} \quad x^2 > 100,$$

and if x is a negative number such that

$$x > -10, \quad \text{then} \quad x^2 < 100.$$

We conclude from this that 10 and -10 are the only elements in the truth set, and that there are at most two elements in the truth set of any sentence of this type. It would appear, then, that the reverse, or as we sometimes say, the inverse of the operation of squaring involves finding the truth set of a special form of sentence.

In the above two examples, we say that 7 and -7 are square roots of 49, and that 10 and -10 are square roots of 100.

Thus the process of reversing the operation of squaring is that of finding the square roots. The question is, "How do we do this?"

In our examples we were able to find the square roots because we know from experience that 49 can be written as 7×7 and that 100 can be written as 10×10. In other words we know that 49 and 100 are squares of integers, and we know what these integers are. In the beginning, then, we can say that in certain cases we find square roots by a process of recognition.

Squares of integers are fairly easy to recognize if they are small enough. We can also find square roots of fractions if both the numerator and denominator are squares of integers. For example we can see that a square root of

$$\frac{16}{25} \text{ is } \frac{4}{5} \text{ because we know that } \frac{4}{5} \cdot \frac{4}{5} = \frac{16}{25}.$$

Do you see that $-\frac{4}{5}$ is also a square root of $\frac{16}{25}$?

However, in the case of larger numbers, we may not be able to recognize the fact that they are squares of integers even though they are. In such cases we will find that factoring is very useful. For example, suppose we are asked to find the square roots of 784. This is the same as the problem of finding the truth set of

$$x^2 = 784.$$

By the methods we used in the previous chapter we can obtain the prime factorization of 784, which is

$$2 \times 2 \times 2 \times 2 \times 7 \times 7.$$

We can group the factors as follows:

$$(2 \times 2 \times 7)(2 \times 2 \times 7)$$

The indicated products in parentheses each represent the same number, 28. This, then, is a square root of 784. Do you see that the other square root is -28?

Check Your Reading

1. What are the square roots of 49?
2. Name the elements in the truth set of "$x^2 = 49$."
3. Why is $\frac{4}{5}$ a square root of $\frac{16}{25}$?
4. How many square roots does $\frac{16}{25}$ have?
5. The prime factorization of 784 is $2 \times 2 \times 2 \times 2 \times 7 \times 7$. How can the factors be "grouped" so as to have two equal factors of 784?
6. What are the square roots of 784?

Oral Exercises 12-1

1. Find the squares of:
    (a) 3
    (b) 4
    (c) 5
    (d) $\frac{1}{2}$
    (e) $\frac{2}{3}$
    (f) $\frac{1}{3}$
    (g) -2
    (h) -7
    (i) -12

Oral Exercises 12-1
(continued)

2. Find the square roots of:

(a) 16
(b) 49
(c) 100
(d) $\frac{4}{9}$
(e) $\frac{1}{25}$
(f) $\frac{9}{16}$
(g) 64
(h) 144
(i) 121

Problem Set 12-1

1. Find the squares of:

(a) 9
(b) -13
(c) $\frac{1}{3}$
(d) 14
(e) -25
(f) $\frac{3}{5}$

2. Find the square roots of:

(a) $\frac{9}{25}$
(b) 36
(c) $\frac{81}{16}$
(d) 169
(e) 225
(f) $\frac{1}{289}$

3. Find the truth sets of the following sentences.

(a) $x^2 = 625$
(b) $m^2 = 324$
(c) $t^2 = \frac{49}{9}$
(d) $\frac{1}{4} = y^2$
(e) $x^2 - 121 = 0$
(f) $y^2 - \frac{16}{81} = 0$

4. Write the prime factorization of each of the following numbers and the prime factorization of its square.

(a) 30
(b) 12
(c) 7
(d) 18
(e) 22
(f) 39

5. Write the prime factorization of each of the following numbers and the prime factorization of its positive square root.

(a) 441
(b) 484
(c) 1764
(d) 676
(e) 784
(f) 15,876

Problem Set 12-1
(continued)

6. If the square of a positive number is decreased by 3, the result is 166. Find the number.

12-2. Radicals.

Thus far we have been finding square roots of numbers which are squares of integers, or fractions whose numerators and denominators are squares of integers. In all cases we have seen that there are two square roots, one positive, and one negative. In every instance the negative square root has been the opposite of the positive square root.

To make things convenient we will use the symbol $\sqrt{\ }$, called a radical sign. It is used to indicate the positive square root of a given real number, if there is such a root. For example,

$$\sqrt{49} = 7 \text{ means}$$

"The positive square root of $\sqrt{49}$ is equal to seven".

We sometimes call $\sqrt{49}$ "radical 49".

In general we say that

$$\sqrt{a} = b, \text{ if } b \text{ is positive, and if } b^2 = a,$$

where a and b are real numbers.

Suppose now that a is a negative real number, say, for example -9. Would the symbol $\sqrt{-9}$ have any meaning? What if we were to say that

$$\sqrt{-9} = b?$$

This would mean that b is a positive real number and that $b^2 = -9$. But whether b is positive or negative, we know that there is no real number whose square is negative. Why? Therefore we can say that $\sqrt{a}$ has no meaning if a is a negative real number.

We shall say that

$$\sqrt{0} = 0 .$$

Check Your Reading

1. Give two different ways in which the symbol "$\sqrt{49}$" might be read.
2. "$\sqrt{49}$" is a numeral for what number?
3. If $\sqrt{a} = b$, what two statements can be made about the number b?
4. Why is it that the symbol "$\sqrt{-9}$" does not represent a real number?
5. State a simpler name for $\sqrt{0}$.

Oral Exercises 12-2

1. Which of the following symbols name a real number and which do not?
   - (a) $\sqrt{36}$
   - (b) $\sqrt{-4}$
   - (c) $\sqrt{81}$
   - (d) $\sqrt{25}$
   - (e) $\sqrt{-25}$
   - (f) $\sqrt{\frac{4}{9}}$
2. In each of the following if the symbol names a real number, state another name for the number.
   - (a) $\sqrt{121}$
   - (b) $\sqrt{9}$
   - (c) $\sqrt{-16}$
   - (d) $\sqrt{\frac{144}{25}}$
   - (e) $\sqrt{256}$
   - (f) $\sqrt{-49}$

Problem Set 12-2

1. In each of the following if the symbol names a real number, write another name for the number.
   - (a) $\sqrt{64}$
   - (b) $\sqrt{\frac{25}{9}}$
   - (c) $\sqrt{(5)(5)}$
   - (d) $\sqrt{-121}$
   - (e) $\sqrt{\frac{49}{169}}$
   - (f) $\sqrt{0}$

Problem Set 12-2
(continued)

2. Find the truth sets of the following:

(a) $x^2 = 16$ (d) $(17)(17) = y^2$

(b) $x = \sqrt{16}$ (e) $t^2 = \frac{625}{9}$

(c) $m^2 = 14$ (f) $t = \sqrt{\frac{625}{9}}$

3. Use prime factorization to write another name for each of the following:

Example:
$$\begin{aligned}\sqrt{441} &= \sqrt{3 \cdot 3 \cdot 7 \cdot 7}\\ &= \sqrt{(3 \cdot 7)(3 \cdot 7)}\\ &= 3 \cdot 7\\ &= 21\end{aligned}$$

(a) $\sqrt{256}$ (d) $\sqrt{1024}$

(b) $\sqrt{1089}$ (e) $\sqrt{576}$

(c) $\sqrt{2304}$ (f) $\sqrt{1936}$

4. (a) The common factors of 3 and 6 are 1 and 3. What are the common factors of 3 and 6 × 6?

(b) The common factors of 2 and 6 are 1 and 2. What are the common factors of 2 and 6 × 6?

(c) What are the common factors of 5 and 10? What are the common factors of 5 and 10 × 10?

(d) What are common factors of 3 and 8? What are the common factors of 3 and 8 × 8?

(e) If the only common factor of b and a is 1, what are the common factors of b and a × a?

12-3. Irrational Numbers.

In finding square roots by the prime factorization method we separated the proper factors into two parts. We then noted that each part contained the same factors as the other one. In finding a square root of 900, for example, we note that the prime factorization is

$$2 \times 2 \times 3 \times 3 \times 5 \times 5.$$

We can group the proper factors as follows:

$$(2 \times 3 \times 5)(2 \times 3 \times 5)$$

It becomes clear that the positive square root is 30.

However, suppose we are asked to find a square root of 84. The prime factorization is

$$2 \times 2 \times 3 \times 7.$$

But this time we cannot separate the factors into two groups which are the same. What then? It might occur to us that we could find another prime factorization which would be more helpful. However, in Chapter 11 we learned that for any given positive integer there is only one prime factorization.

From this we see that there is no positive integer which is the square root of 84. The same thing is true of 12 whose prime factorization is

$$2 \times 3 \times 2,$$

and 10 whose prime factorization is

$$2 \times 5.$$

We must now ask a very important question. Since there is no positive integer which is a square root of, say, 10, is there a rational number, not an integer, whose square is 10? In other words, can we find a rational number $\frac{a}{b}$ whose square is 10? This is a hard question to answer. We know that $3^2 = 9$, and $4^2 = 16$, so 3 is too small, and 4 is too large. Therefore our rational number, whatever it is, must be somewhere between 3 and 4. Remember that a rational number is one which we must be able to write as a fraction whose numerator and denominator are both integers. A rational number between 3 and 4 which might work would be $3\frac{1}{4}$, which we can write as

$$\frac{13}{4}.$$

However, we see that

$$\frac{13}{4} \cdot \frac{13}{4} = \frac{169}{16}$$

which is a little larger than 10. Do you agree?

We could try some others. But remember that in Chapter 1 we learned by means of the number line that between any two rational numbers there are infinitely many others. In other words we could keep on trying for a long time and our search might never end!

But here is where mathematics comes to the rescue. We are about to show that there is *no* rational number which is a square root of 10. Therefore, we don't need to hunt any further.

In showing this, we shall also prove a theorem which says a great deal more. This theorem may be stated as follows:

> If any integer has a square root which is rational, then the square root *is also an integer*.

For example, the number 10 is an integer. The theorem says that if 10 has a square root that is rational, then the square root must also be an integer, such as 3 or 4. But we already know that neither of these works; we also know that any other integer will be either too small or too large. Thus, once the theorem is proved, we will be able to say that 10---as well as many other numbers---has *no rational square roots*. First, however, we must prove the theorem; the proof follows.

We begin by picking any integer and calling this integer n. We then suppose that n has a square root which is a rational number and which we call $\frac{a}{b}$. We also say that $\frac{a}{b}$ is written in simplest form. That is, $\frac{a}{b}$ is the common name for our rational number. We learned in Chapter 10 what this means. It means that a and b have no common factor except 1.

Since $\frac{a}{b}$ is a square root of n, then

$$\frac{a}{b} \cdot \frac{a}{b} = n; \text{ that is, } \frac{a^2}{b^2} = n.$$

We know that b is not zero. Why? (Remember the description of a rational number in Chapter 1.) Therefore we can multiply both sides of our sentence by b^2 and obtain

$$a \times a = b \times b \times n.$$

This means that b is a factor of the integer $a \times a$. In other words all the prime factors of b are contained among the prime factors of $(a \times a)$. But $(a \times a)$ has no prime factors which are different from those of a. Each prime factor of a merely appears twice as often in the prime factorization of $(a \times a)$.

Therefore, in looking at b we see that either b has a prime factor in common with a, or else b is 1.

But we started by saying that a and b have no common proper factor. Hence b must be 1.

Thus we have shown that our rational number $\frac{a}{b}$ must be an integer.

We can now use this very important idea to find out whether or not any integer has a square root which is a rational number. We can already see that 10 has no rational square root. Suppose we try now the number 30. Does 30 have a rational square root? We are asking, in other words, whether or not

$$\sqrt{30}$$

represents a rational number. We can easily check this. By the argument above, we see that if there is a rational square root of 30, then it must be an integer. We try 5, and we try 6. But

$$5^2 = 25 \text{ and } 6^2 = 36.$$

Five is too small and six is too large. There is no integer between 5 and 6. Therefore 30 does not have a square root which is a rational number.

We can also check whether an integer has an integral square root by prime factorization: if the prime factors do not come in pairs, then there is no integral square root, therefore no rational square root.

Check Your Reading

1. The prime factorization of 900 is $2 \times 2 \times 3 \times 3 \times 5 \times 5$. Can these factors be separated into two "groups" which are the same?
2. What is the positive square root of 900?
3. The prime factorization of 84 is $2 \times 2 \times 3 \times 7$. Can these factors be separated into two "groups" which are the same?
4. What is 3^2? What is 4^2? The square root of 10 lies between what two integers?
5. Is the positive square root of 10 a rational number between 3 and 4?
6. Complete the statement of the following theorem, which was developed in this section of the text:

   If an integer has a square root which is a
   rational number, then . . .
7. Does 30 have a rational square root? Why or why not?

Oral Exercises 12-3a

For each of the following integers, state its positive rational square root if there is one. If not, show why not.

Example: 5.

Solution: If the integer 5 has a rational square root, it must be an integer. But 2 is too small since $2^2 = 4$ and 3 is too large since $3^2 = 9$. There are no integers between 2 and 3; hence, 5 has no rational square root.

Oral Exercises 12-3a
(continued)

1. 3
2. 4
3. 5
4. 6
5. 9
6. 16
7. 13
8. 25
9. 17
10. 18

Problem Set 12-3a

1. For each of the following integers, state its positive rational square root if there is one. If not, show why not.

   (a) 11
   (b) 36
   (c) 49
   (d) 23
   (e) 35
   (f) 625

2. For each of the following integers, state its positive rational square root if there is one. If not, show why not.

   Example 1. $3^2 \cdot 5^2$

   Solution: Since $(3 \cdot 5)^2 = 3^2 \cdot 5^2$; $3 \cdot 5$ is the required square root.

   Example 2. $2^2 \cdot 3$

   Solution: If the integer $2^2 \cdot 3$ has a rational square root, it must be an integer. But the prime factors do not occur in pairs. Therefore there is no integer whose square is $2^2 \cdot 3$.

   (a) $2 \times 2 \times 2 \times 2$
   (b) $3 \times 3 \times 5$
   (c) $3^2 \cdot 7^2$
   (d) $2^2 \cdot 3^2 \cdot 7^2 \cdot 11^2$
   (e) $3^2 \cdot 5^2 \cdot 7$
   (f) $2^4 \cdot 13^2$

Problem Set 12-3a
(continued)

3. For each of the following integers, state its positive rational square root if there is one.

   (a) 144
   (b) $5^2 \cdot 7^2 \cdot 23^2$
   (c) 117
   (d) 2448 (Hint: prime factorization)
   (e) 592
   (f) 941

4. For each of the following integers, state its positive rational square root if there is one.

   (a) 1025
   (b) 252
   (c) 75
   (d) 37^2
   (e) 675
   (f) 900

We have shown that a symbol such as $\sqrt{30}$ does not represent a rational number. The same can be said for $\sqrt{2}$.

It is important now to review some important ideas about real numbers. In Chapter 6 it was stated that the *real* numbers are those numbers that can be associated with points of the real number line. It was also stated that they include *rational* numbers and *irrational* numbers.

It is possible to show that $\sqrt{2}$ is the coordinate of a point on the number line. The same is true of $\sqrt{3}$, $\sqrt{5}$, and others. Therefore these are real numbers. But we have just learned that they are not rational numbers. Therefore we can see that there are real numbers which are not rational numbers. We call such numbers

irrational numbers.

Thus we see that the set of real numbers consists of two subsets, the rational numbers and the irrational numbers. Every real number is an element of exactly one of these two sets.

Since an irrational number cannot be written as the quotient of two integers, we need to use the radical sign a great deal. We have already shown that a numeral such as "$\sqrt{25}$"

can be written in the simpler form "5". However, in the case of a square root of 2, the simplest form we can write this in is

$$\sqrt{2}\ .$$

You recall that this represents the positive square root of 2. For the negative square root of 2 we write

$$-\sqrt{2}\ .$$

Check Your Reading

1. Complete this statement: Real numbers are those numbers that can be associated with points of the . . .
2. Complete this statement: Some real numbers are rational numbers; all of the others are . . .
3. Which of the following are irrational numbers: $\sqrt{2}$, $\sqrt{25}$, 5 ?
4. Write numerals for the two square roots of 2.

Oral Exercises 12-3b

1. Which of the following are rational and which are irrational?

   (a) $\sqrt{2}$ (d) $\sqrt{5}$ (g) $\sqrt{8}$
   (b) $\sqrt{3}$ (e) $\sqrt{6}$ (h) $\sqrt{9}$
   (c) $\sqrt{4}$ (f) $\sqrt{7}$ (i) $\sqrt{10}$

2. Find the truth sets of each of the following:

   (a) $x^2 = 2$ (d) $x^2 = 5$ (g) $x^2 = 8$
   (b) $x^2 = 3$ (e) $x^2 = 6$ (h) $x^2 = 9$
   (c) $x^2 = 4$ (f) $x^2 = 7$ (i) $x^2 = 10$

Problem Set 12-3b

1. Which of the following are rational and which are irrational?

   (a) $\sqrt{16}$ (d) $-\sqrt{49}$ (g) $\frac{1}{\sqrt{9}}$
   (b) $\sqrt{21}$ (e) $\sqrt{50}$
   (c) $(\sqrt{3})^2$ (f) $(-\sqrt{5})^2$

Problem Set 12-3b
(continued)

2. Find the truth sets of:

   (a) $x^2 = 5$ (d) $z^2 = -3$

   (b) $y^2 = 7$ (e) $t = \sqrt{36}$

   (c) $81 = m^2$ (f) $x = \sqrt{11}$

3. Find the truth sets of:

   (a) $3x^2 = 15$ (d) $7m^2 - 4 = 17$

   (b) $75 = 3y^2$ (e) $x^2 = 11$ and $x < 0$

   (c) $14 + 2z^2 = 18$ (f) $3t^2 = (\sqrt{3})^2$

4. Find the truth sets of:

   (a) $8k^2 = 16$

   (b) $16x^2 = 64$

   (c) $z = \sqrt{49}$

   (d) $5m^2 = 15$

   (e) $x = \sqrt{3}$ or $x = -\sqrt{3}$ (Compound sentence)

   (f) $x^2 + 2 = 0$ or $x^2 - 3 = 0$ (Compound sentence)

5. If m and k are primes and

   $$1 \cdot 2 \cdot m \cdot 5^2 = 1 \cdot k \cdot 3 \cdot 5^2,$$

   What are the values of m and k?

12-4. Simplification of Radicals.

In the previous section we observed that the simplest form in which we can write the positive square root of 2 is

$$\sqrt{2}.$$

However, there are radicals which can be simplified, that is, written in a simpler form even though they are not squares of integers. Examples of such radicals are

$$\sqrt{12} \qquad \sqrt{18} \qquad \sqrt{50} \qquad \sqrt{x^3}\,,\quad x \text{ non-negative}$$

To do this we will need an important property of radicals.

First consider the following problem:

Write the indicated product, $(\sqrt{4})(\sqrt{9})$, as a single radical.

We can see that $\sqrt{4} = 2$, and $\sqrt{9} = 3$. It follows that

$$\begin{aligned}(\sqrt{4})(\sqrt{9}) &= 2 \cdot 3 \\ &= 6.\end{aligned}$$

But we also know that

$$6 = \sqrt{36}.$$

Using the transitive property, we can see that

$(\sqrt{4})(\sqrt{9})$ must be equal to $\sqrt{36}$,

which is the same as saying that

$$(\sqrt{4})(\sqrt{9}) = \sqrt{4 \cdot 9}.$$

In this case we have found how the product of two radicals can be written as a single radical. But remember, we could do this because the numerals under the radicals represented squares of integers. In other words, $\sqrt{4}$ and $\sqrt{9}$ are symbols for the integers 2 and 3, which we already know how to multiply.

However, suppose we are given the problem of writing a simpler form for

$$(\sqrt{3})(\sqrt{5}).$$

Do we know that this can be written as

$$\sqrt{15}\ ?$$

We cannot answer as we answered before, since these are irrational numbers. Therefore we must try a different approach. For the real numbers $\sqrt{3}$ and $\sqrt{5}$ we know that

$$\begin{aligned}\left((\sqrt{3})(\sqrt{5})\right)^2 &= (\sqrt{3})^2(\sqrt{5})^2 \\ &= 3 \cdot 5 \\ &= 15.\end{aligned}$$

This is another way of saying that $(\sqrt{3})(\sqrt{5})$ is a square root of 15.
We know that $(\sqrt{3})(\sqrt{5})$ is positive.
Therefore, we can write

$$(\sqrt{3})(\sqrt{5}) = \sqrt{15} .$$

It should be clear that the above would hold for any non-negative real numbers as well as 3 and 5. Thus we can state the following general property.

For any two non-negative real numbers a and b

$$(\sqrt{a})(\sqrt{b}) = \sqrt{ab} .$$

This tells us, for example, that the following sentences are true.

$$(\sqrt{7})(\sqrt{11}) = \sqrt{77}, \quad \left(\sqrt{\tfrac{2}{3}}\right)\left(\sqrt{\tfrac{3}{5}}\right) = \sqrt{\tfrac{2}{5}}, \quad \text{and}$$

$$\sqrt{3}\,(\sqrt{5} + \sqrt{2}) = \sqrt{15} + \sqrt{6}.$$

In the last sentence we have used the distributive property, which, as you remember applies to all real numbers.

Check Your Reading

1. The fact that "$(\sqrt{4})(\sqrt{9}) = 6$" shows that $(\sqrt{4})(\sqrt{9})$ is the positive square root of what number?
2. Give another name for the product $(\sqrt{3})(\sqrt{5})$.
3. Provided that a and b are non-negative, give another name for the product $(\sqrt{a})(\sqrt{b})$.
4. Use the distributive property to write the indicated product $\sqrt{3}(\sqrt{5} + \sqrt{2})$ as an indicated sum.

Oral Exercises 12-4a

Multiply using $\sqrt{a}\sqrt{b} = \sqrt{ab}$. Use the distributive property where it applies.

1. $\sqrt{2}\sqrt{2}$
2. $\sqrt{2}\sqrt{3}$
3. $\sqrt{2}\sqrt{8}$
4. $\sqrt{3}\sqrt{6}$
5. $\sqrt{2}\,(\sqrt{2} + \sqrt{3})$
6. $\sqrt{2}\,(\sqrt{2} + \sqrt{8})$
7. $\sqrt{3}\sqrt{12}$
8. $\sqrt{3}\,(\sqrt{6} + \sqrt{12})$
9. $\sqrt{5}\sqrt{7}$
10. $\sqrt{2}\sqrt{3}\sqrt{3}$

Problem Set 12-4a

1. Simplify each of the following products.
   - (a) $\sqrt{2}\sqrt{18}$
   - (b) $\sqrt{5}\sqrt{6}$
   - (c) $\sqrt{3}\sqrt{3}$
   - (d) $\sqrt{3}\sqrt{27}$
   - (e) $\sqrt{5}\sqrt{12}$
   - (f) $\sqrt{3}\sqrt{75}$

2. Simplify each of the following products.
   - (a) $\sqrt{2}\,(\sqrt{3} + \sqrt{8})$
   - (b) $\sqrt{5}\,(\sqrt{5} + \sqrt{6})$
   - (c) $(\sqrt{3} + \sqrt{12})\sqrt{3}$
   - (d) $\sqrt{2}\sqrt{0}$
   - (e) $\sqrt{3}\sqrt{5}\sqrt{7}$
   - (f) $\sqrt{2}\sqrt{3}\sqrt{2}$

3. Simplify each of the following products. In those problems involving variables indicate what restrictions must be put on the domains of the variables.
   - (a) $\sqrt{2}\sqrt{x}$
   - (b) $\sqrt{2}\sqrt{8y}$
   - (c) $\sqrt{3a}\sqrt{12}$
   - (d) $\sqrt{2}\,(\sqrt{2x} + \sqrt{8})$
   - (e) $m\sqrt{2}\sqrt{18}$
   - (f) $\sqrt{y}\sqrt{y}$

Problem Set 12-4a
(continued)

4. Find the truth sets of the following open sentences.

(a) $\frac{1}{\sqrt{2}}x = \sqrt{8}$ (d) $\frac{m}{\sqrt{2}} = \sqrt{32} + \sqrt{8}$

(b) $\frac{1}{\sqrt{3}}y = \sqrt{12}$ (e) $\frac{1}{\sqrt{3}}t = \sqrt{27}$

(c) $\frac{1}{\sqrt{2}}z = \sqrt{3}$ (f) $\frac{3x^2}{\sqrt{18}} = \sqrt{2}$

5. Multiply, indicating the restrictions on the domains of the variables.

(a) $(\sqrt{2} + 1)(\sqrt{2} - 1)$ (d) $\sqrt{x}\,(\sqrt{x} + 1)$

(b) $(\sqrt{2} + \sqrt{3})(\sqrt{2} - \sqrt{3})$ (e) $(\sqrt{y} + 1)(\sqrt{y} - 1)$

(c) $(\sqrt{2} + 1)^2$ (f) $(\sqrt{a} + \sqrt{b})^2$

From the following examples we will see how to simplify certain forms of radicals. Suppose we wish to write a simpler form for the numeral

$$\sqrt{12}.$$

Our knowledge of factoring enables us to write this as

$$\sqrt{2 \cdot 2 \cdot 3}.$$

The property we have just studied enables us to write

$$\begin{aligned}\sqrt{2 \cdot 2 \cdot 3} &= \sqrt{2 \cdot 2}\ \sqrt{3} \\ &= \sqrt{2^2}\ \sqrt{3}\ .\end{aligned}$$

But we know that $\sqrt{2^2} = 2$. Therefore our final form is

$$2\sqrt{3}\ .$$

We call this the simplest form of $\sqrt{12}$. It is the form in which the smallest possible integer remains under the radical sign.

In the next example we apply our property to radicals containing a larger number of factors. We can do this because of the associative law of multiplication. Suppose we are asked to simplify, that is, find the simplest form of $\sqrt{180}$. We write

$$\begin{aligned}\sqrt{180} &= \sqrt{2^2 \cdot 3^2 \cdot 5} \\ &= \sqrt{2^2}\,\sqrt{3^2}\,\sqrt{5} \\ &= (2 \cdot 3)\sqrt{5} \\ &= 6\sqrt{5}\,.\end{aligned}$$

This last is the simplest form of $\sqrt{180}$.

In the above examples we have observed that expressions such as 2^2 and 3^2 are clearly the squares of integers; hence $\sqrt{2^2}$ and $\sqrt{3^2}$ may be represented as 2 and 3 without the radical sign. It should be clear that this can always be done if the exponent is an even number. For example, 5^4 is the square of the integer 5^2 which is 25.

Do you see that

$$\sqrt{7^6} = 7^3 \quad \text{and} \quad \sqrt{3^{10}} = 3^5 \text{ ?}$$

If we are working with radicals which contain variables, we must pay special attention to the following idea. Suppose we are asked to simplify the radical

$$\sqrt{x^2}.$$

We know that this represents a real number for all values of x. Why? It would seem natural, then, to write the following:

$$\sqrt{x^2} = x.$$

But now we must ask the question, "Is our sentence true for all values of x?" Suppose x has the value -7. Our sentence then becomes:

$$\sqrt{49} = -7 \text{ which is false. Why?}$$

Because the radical always represents a non-negative square root. To avoid this difficulty we always write in such cases

$$\sqrt{x^2} = |x|\ .$$

Do you see why this sentence is true for all values of x? Review the meaning of $|x|$ in Chapter 6.

Here is a somewhat more complicated example.

$$\begin{aligned}\sqrt{5x^2y^3} &= \sqrt{5x^2y^2y}\\ &= \sqrt{x^2y^2}\cdot\sqrt{5y}\\ &= \sqrt{x^2}\,\sqrt{y^2}\cdot\sqrt{5y}\\ &= |x|\,|y|\cdot\sqrt{5y}\end{aligned}$$

It might be interesting to note that if we were asked to simplify $\sqrt{y^4}$, we could write

$$\sqrt{y^4} = |y^2| .$$

However in this case it would not be necessary to write the absolute value symbol for y^2. Do you see why? There is no value of y for which y^2 is negative.

Check Your Reading

1. "$\sqrt{12}$" and "$2\sqrt{3}$" are names for the same number. Which is considered simpler?
2. Give a simpler name for the number $\sqrt{2^2}$.
3. Give a simpler name for the number $\sqrt{(-7)^2}$.
4. Describe the truth set of "$\sqrt{x^2} = x$."
5. Describe the truth set of "$\sqrt{x^2} = |x|$."
6. Give a simpler name for $\sqrt{5x^2y^3}$.
7. Give a simpler name for $\sqrt{y^4}$.

Oral Exercises 12-4b

Simplify the following radicals; indicate the restrictions on the domain of the variables.

1. $\sqrt{7^2}$
2. $\sqrt{5^2\cdot 3}$
3. $\sqrt{3^4\cdot 2}$
4. $\sqrt{3^6}$

81

Oral Exercises 12-4b
(continued)

5. $\sqrt{2^{10}}$
6. $\sqrt{11^2 \cdot 11}$
7. $\sqrt{11^3}$
8. $\sqrt{5^3}$
9. $\sqrt{50}$
10. $\sqrt{18}$
11. $\sqrt{27}$
12. $\sqrt{a^2}$
13. $\sqrt{m^2}$
14. $\sqrt{9b^2}$
15. $\sqrt{16t^2}$
16. $\sqrt{t^3}$

Problem Set 12-4b

1. Simplify:
   (a) $\sqrt{20}$
   (b) $\sqrt{32}$
   (c) $\sqrt{24}$
   (d) $\sqrt{150}$
   (e) $\sqrt{48}$
   (f) $\sqrt{45}$

2. Simplify:
   (a) $2\sqrt{12}$
   (b) $3\sqrt{36}$
   (c) $\sqrt{96}$
   (d) $\sqrt{108}$
   (e) $2\sqrt{18}$
   (f) $3\sqrt{121}$

3. Simplify:
   (a) $\sqrt{30}$
   (b) $\sqrt{12}\sqrt{6}$
   (c) $\sqrt{2}\sqrt{10}$
   (d) $\sqrt{252}$
   (e) $\sqrt{2}\sqrt{14}$
   (f) $\sqrt{5}\sqrt{20}$

4. Simplify; and indicate the restrictions on the variable.
   (a) $\sqrt{24x^2}$
   (b) $\sqrt{24x^3}$
   (c) $\sqrt{24x^4}$
   (d) $\sqrt{32a}$
   (e) $\sqrt{32a^2}$
   (f) $\sqrt{32a^3}$

Problem Set 12-4b
(continued)

5. Find the truth sets of the following, writing each radical in simplest form.

   (a) $x^2 = 56$

   (b) $2y^2 = 324$

   (c) $5m^2 = 900$

   (d) $\frac{1}{3}t^2 = 16$

   (e) $49 = \frac{1}{11}n^2$

   (f) $\sqrt{2} = \frac{x}{\sqrt{3}}$

6. Simplify, and indicate the restrictions on the variable.

   (a) $\sqrt{25x^3}$

   (b) $\sqrt{3x}\sqrt{6x}$

   (c) $\sqrt{5x^7}$

   (d) $\sqrt{625y^2}$

   (e) $\sqrt{196m}$

   (f) $2\sqrt{3x}\sqrt{4x}$

12-5. Simplification of Radicals Involving Fractions.

Thus far we have been simplifying radicals which for the most part have contained positive integers. By means of a second property for radicals we can work with fractions in much the same way. To show how this property works we will begin with an example.

We have seen that

$$\sqrt{\frac{16}{25}} = \sqrt{\frac{4}{5}}.$$

It should also be clear that

$$\frac{\sqrt{16}}{\sqrt{25}} = \frac{4}{5}, \quad \text{hence} \sqrt{\frac{16}{25}} = \frac{\sqrt{16}}{\sqrt{25}}.$$

We see in this case that a single radical containing a fraction can be written as the quotient of two radicals. Is this true if the numerator and denominator are not squares of integers?

For example, can $\sqrt{\frac{3}{5}}$ be written as $\frac{\sqrt{3}}{\sqrt{5}}$?

We do know that

$$\left(\frac{\sqrt{3}}{\sqrt{5}}\right)^2 = \left(\frac{\sqrt{3}}{\sqrt{5}}\right)\left(\frac{\sqrt{3}}{\sqrt{5}}\right)$$

$$= \frac{(\sqrt{3})^2}{(\sqrt{5})^2}$$

$$= \frac{3}{5}.$$

This tells us that $\frac{\sqrt{3}}{\sqrt{5}}$ is a square root of $\frac{3}{5}$. Since $\frac{\sqrt{3}}{\sqrt{5}}$ is positive, we can say that $\frac{\sqrt{3}}{\sqrt{5}} = \sqrt{\frac{3}{5}}$.

The general property can be stated as follows: If a is a non-negative real number, and if b is a positive real number, then

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}.$$

With this property in mind we can now simplify an expression like $\sqrt{\frac{8}{9}}$. We say

$$\sqrt{\frac{8}{9}} = \frac{\sqrt{8}}{\sqrt{9}}$$

$$= \frac{\sqrt{2 \cdot 2 \cdot 2}}{\sqrt{3 \cdot 3}}$$

$$= \frac{2\sqrt{2}}{3},$$ which can be written $\frac{2}{3}\sqrt{2}$.

A second example involving a variable is

$$\sqrt{\frac{15}{5x^2}} = \frac{\sqrt{3 \cdot 5}}{\sqrt{5 \cdot x^2}}$$

$$= \frac{\sqrt{3} \cdot \sqrt{5}}{|x| \cdot \sqrt{5}}$$

$$= \frac{\sqrt{3}}{|x|}, \quad x \neq 0$$

The property can be used "in reverse". For example, if we are asked to simplify an expression such as

$$\frac{\sqrt{18}}{\sqrt{2}},$$

we can see that

$$\begin{aligned}\frac{\sqrt{18}}{\sqrt{2}} &= \sqrt{\frac{18}{2}}\\ &= \sqrt{9}\\ &= 3\,.\end{aligned}$$

Check Your Reading

1. Express $\sqrt{\frac{16}{25}}$ as the quotient of two radicals.
2. Express $\frac{\sqrt{3}}{\sqrt{5}}$ as a single radical.
3. Complete this sentence: For any non-negative a and positive b, $\sqrt{\frac{a}{b}}$ = _______.
4. A simpler phrase for $\sqrt{\frac{15}{5x^2}}$ is _______.
5. A simpler phrase for $\frac{\sqrt{18}}{\sqrt{2}}$ is _______.

Oral Exercises 12-5a

Simplify:

1. $\sqrt{\frac{4}{9}}$
2. $\sqrt{\frac{32}{25}}$
3. $\sqrt{\frac{144}{49}}$
4. $\sqrt{\frac{75}{121}}$
5. $\sqrt{\frac{8}{2}}$
6. $\sqrt{\frac{50}{2}}$
7. $\sqrt{\frac{24}{3}}$
8. $\sqrt{\frac{27}{3}}$
9. $\sqrt{\frac{15}{5}}$
10. $\sqrt{\frac{5^2 \cdot 7^2}{13^2}}$

Problem Set 12-5a

For the following problems indicate the restrictions on the variable.

1. Simplify:

(a) $\sqrt{\frac{16}{25}}$

(b) $\sqrt{\frac{49}{64}}$

(c) $\sqrt{\frac{27}{16}}$

(d) $\sqrt{\frac{16x^2}{9}}$

(e) $\sqrt{\frac{8}{9x^2}}$

(f) $\sqrt{\frac{3^4 \cdot 5^2}{7^2}}$

2. Simplify:

(a) $\frac{\sqrt{24}}{\sqrt{8}}$

(b) $\frac{\sqrt{21}}{\sqrt{3}}$

(c) $\frac{\sqrt{66}}{\sqrt{11}}$

(d) $\frac{\sqrt{15x^2}}{\sqrt{5}}$

(e) $\frac{\sqrt{60m}}{\sqrt{12m}}$

(f) $\frac{\sqrt{7^3 \cdot 11^2}}{\sqrt{7}}$

3. Simplify:

(a) $\sqrt{\frac{12}{25}}$

(b) $\sqrt{\frac{49}{a^2}}$

(c) $\frac{\sqrt{441}}{\sqrt{3}}$

(d) $\sqrt{\frac{x^3}{y^2}}$

(e) $\frac{\sqrt{x^3}}{\sqrt{x^2}}$

(f) $\frac{\sqrt{392a^3}}{\sqrt{2a}}$

4. Simplify:

(a) $\frac{\sqrt{2}\ \sqrt{6}}{\sqrt{3}}$

(b) $\frac{\sqrt{2}\ \sqrt{3}\ \sqrt{5}}{\sqrt{6}}$

(c) $\sqrt{\frac{6}{27a^2}}$

(d) $\frac{y\sqrt{y}}{\sqrt{y^3}}$

(e) $\frac{\sqrt{252}}{\sqrt{144}}$

(f) $\sqrt{\frac{75a^3}{81y^2}}$

Problem Set 12-5a
(continued)

5. Find the truth sets of the following open sentences.

(a) $\sqrt{2}\,x = \sqrt{18}$

(b) $\sqrt{3}\,y = \sqrt{24}$

(c) $\sqrt{2} + \sqrt{5}\,t = \sqrt{35} + \sqrt{2}$

(d) $\dfrac{t}{\sqrt{2}} = \dfrac{\sqrt{6}}{\sqrt{3}}$

(e) $\dfrac{\sqrt{2}}{\sqrt{72}} = \dfrac{1}{x}$

(f) $\sqrt{2}\,(y + 3) = \sqrt{72} + \sqrt{6}$

We come now to the case of a radical, like $\sqrt{\frac{3}{5}}$. containing a quotient of two integers in which the denominator is not the square of an integer. To simplify such a radical will mean to write it in a form in which the numerator or denominator is free of radicals. As an example, let us find two simpler forms for $\sqrt{\frac{3}{5}}$.

We know we can write

$$\sqrt{\frac{3}{5}} = \frac{\sqrt{3}}{\sqrt{5}}.$$

If we multiply by

$$\frac{\sqrt{3}}{\sqrt{3}}, \quad \text{which is another name for } 1,$$

we get

$$\frac{\sqrt{3}}{\sqrt{5}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3 \cdot 3}}{\sqrt{5 \cdot 3}}$$
$$= \frac{3}{\sqrt{15}}$$

Another simplified form is obtained by multiplying $\frac{\sqrt{3}}{\sqrt{5}}$ by $\frac{\sqrt{5}}{\sqrt{5}}$ which is another name for 1.

This will give us

$$\frac{\sqrt{3}}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{15}}{5}.$$

Both $\frac{3}{\sqrt{15}}$ and $\frac{\sqrt{15}}{5}$ are simpler forms than the one we started with. Later we will show that the second form is more convenient for some purposes. We call the first process "rationalizing the numerator," since the new form $\frac{3}{\sqrt{15}}$ gives a numerator which is a rational number. For the same reason, the second process is called "rationalizing the denominator".

Example 1. Simplify $\sqrt{\frac{7}{12}}$ by the process of rationalizing the denominator.

$$\sqrt{\frac{7}{12}} = \frac{\sqrt{7}}{\sqrt{12}}$$
$$= \frac{\sqrt{7}}{2\sqrt{3}}$$

Multiplying by $\frac{\sqrt{3}}{\sqrt{3}}$ we get

$$\frac{\sqrt{7}}{2\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{21}}{2 \cdot 3}$$
$$= \frac{\sqrt{21}}{6}.$$

Example 2. Simplify $\sqrt{\frac{3}{2x^2}}$ by the process of rationalizing the denominator.

The steps are $\sqrt{\frac{3}{2x^2}} = \frac{\sqrt{3}}{\sqrt{2x^2}}, \quad (x \neq 0)$

$$= \frac{\sqrt{3}}{|x|\sqrt{2}}.$$

We multiply by $\frac{\sqrt{2}}{\sqrt{2}}$ and get $\frac{\sqrt{3}}{|x|\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{6}}{2|x|}$, $(x \neq 0)$.

Example 3. Simplify $\sqrt{\frac{x}{2}}$ by the process of rationalizing the numerator.

In this case we see the need for restricting the domain of the variable. The expression $\sqrt{\frac{x}{2}}$ is a real number only if x is non-negative. But we wish to multiply by $\frac{\sqrt{x}}{\sqrt{x}}$. Therefore we must say that x cannot be negative or zero. Do you see why? Our example can be written as follows:

$$\begin{aligned}\sqrt{\frac{x}{2}} &= \frac{\sqrt{x}}{\sqrt{2}}, \qquad x > 0\\ &= \frac{\sqrt{x}}{\sqrt{2}} \cdot \frac{\sqrt{x}}{\sqrt{x}}, \qquad x > 0\\ &= \frac{x}{\sqrt{2x}}, \qquad x > 0.\end{aligned}$$

Check Your Reading

1. What name for one do we use in rationalizing the denominator of $\frac{\sqrt{3}}{\sqrt{5}}$?
2. What name for one do we use in rationalizing the numerator of $\frac{\sqrt{3}}{\sqrt{5}}$?
3. Why is it that in the expression $\sqrt{\frac{x}{2}}$ the variable must be non-negative?

Oral Exercises 12-5b

1. Rationalize the denominator.
   - (a) $\sqrt{\frac{1}{3}}$
   - (b) $\sqrt{\frac{1}{2}}$
   - (c) $\frac{1}{\sqrt{5}}$
   - (d) $\frac{\sqrt{3}}{\sqrt{7}}$
   - (e) $\sqrt{\frac{3}{5}}$
   - (f) $\sqrt{\frac{10}{3}}$

2. Rationalize the numerator.
   - (a) $\sqrt{\frac{2}{3}}$
   - (b) $\frac{\sqrt{5}}{\sqrt{3}}$
   - (c) $\frac{\sqrt{2}}{2}$
   - (d) $\frac{\sqrt{3}}{3}$
   - (e) $\sqrt{\frac{2}{7}}$
   - (f) $\frac{\sqrt{7}}{\sqrt{5}}$

Problem Set 12-5b

1. Rationalize the denominator.

(a) $\frac{2}{\sqrt{2}}$

(b) $\frac{1}{\sqrt{11}}$

(c) $\frac{3}{\sqrt{3}}$

(d) $\sqrt{\frac{2}{7}}$

(e) $\sqrt{\frac{4}{5}}$

(f) $\frac{1}{\sqrt{8}}$

2. Rationalize the numerator.

(a) $\sqrt{\frac{6}{7}}$

(b) $\frac{\sqrt{5}}{\sqrt{11}}$

(c) $\frac{\sqrt{3}}{3}$

(d) $\frac{\sqrt{2}}{2}$

(e) $\frac{\sqrt{8}}{\sqrt{3}}$

(f) $\sqrt{\frac{12}{5}}$

3. Simplify by the process of rationalizing the denominator. Indicate the restrictions on the variable.

(a) $\sqrt{\frac{9}{50}}$

(b) $3\sqrt{\frac{7}{36}}$

(c) $\sqrt{\frac{75}{63}}$

(d) $\sqrt{\frac{3b}{5}}$

(e) $\sqrt{\frac{5}{x}}$

(f) $\sqrt{\frac{2}{7b}}$

4. Simplify by the process of rationalizing the numerator. Indicate the restrictions on the variables.

(a) $\frac{1}{4}\frac{\sqrt{16}}{\sqrt{5}}$

(b) $\frac{2\sqrt{8}}{\sqrt{9}}$

(c) $\sqrt{\frac{75}{63}}$

(d) $\frac{\sqrt{m}}{\sqrt{3}}$

(e) $\sqrt{\frac{t^3}{2}}$

(f) $\sqrt{\frac{2y^2}{3}}$

Problem Set 12-5b
(continued)

5. Simplify by the process of rationalizing the denominator.

(a) $\dfrac{1}{\sqrt{2^3 \cdot 3^3}}$ (d) $\dfrac{1}{\sqrt{180}}$

(b) $\dfrac{\sqrt{7 \cdot 3^2}}{\sqrt{5^3 \cdot 2}}$ (e) $\sqrt{\dfrac{75}{768}}$

(c) $\dfrac{\sqrt{2^2 \cdot 3^2 \cdot 5^3}}{\sqrt{5^2 \cdot 7^3 \cdot 11}}$ (f) $\sqrt{\dfrac{512}{243}}$

12-6. Sums and Differences of Radicals.

We have seen what can be done with products and quotients of radicals in order to simplify them. We will now consider sums and differences of radicals. We will see that in this situation there is often nothing we can do to simplify a given expression. Suppose we are given, for example, the phrase

$$\sqrt{2} + \sqrt{3} .$$

It is clear that $\sqrt{2}$ and $\sqrt{3}$ are each in simplest form, and there is no way in which we can perform the indicated operation of addition. Therefore we say that the phrase

$$\sqrt{2} + \sqrt{3}$$

is already in simplest form.
On the other hand, suppose we are given an expression such as

$$4\sqrt{3} + 3\sqrt{12} .$$

We see that the expression on the right may be simplified as follows:

$$\begin{aligned} 3\sqrt{12} &= 3\sqrt{2 \cdot 2 \cdot 3} \\ &= 3 \cdot 2\sqrt{3} \\ &= 6\sqrt{3} . \end{aligned}$$

Thus our original expression may be written as

$$4\sqrt{3} + 6\sqrt{3} .$$

By the distributive property

$$4 \cdot \sqrt{3} + 6 \cdot \sqrt{3} = (4 + 6) \cdot \sqrt{3},$$

which may be written as

$$10 \cdot \sqrt{3}.$$

This form is the simplest one.

The difference of two radicals can often be simplified in much the same way. We try first to work with each radical separately to see if it can be simplified. If our expression can be written so that both radicals are the same, we may then combine the two parts by the distributive property as above.

As an example, simplify

$$\sqrt{63} - \sqrt{28}.$$

The radical on the left can be written as $3\sqrt{7}$.

The radical on the right can be written as $2\sqrt{7}$. Do you see why? This gives us

$$3\sqrt{7} - 2\sqrt{7}.$$

Using the distributive property we obtain $(3 - 2)\sqrt{7} = \sqrt{7}$.

Check Your Reading

1. Give a simpler name for the number $4\sqrt{3} + 6\sqrt{3}$.
2. What property of the real numbers is used in simplifying $4\sqrt{3} + 6\sqrt{3}$?
3. Give a simpler name for $3\sqrt{7} - 2\sqrt{7}$.
4. Give a simpler name for $\sqrt{63} - \sqrt{28}$.

Oral Exercises 12-6

Simplify. Indicate the restrictions on the variables.

1. $2\sqrt{2} + 3\sqrt{2}$
2. $5\sqrt{3} - 2\sqrt{3}$
3. $7\sqrt{5} + \sqrt{5}$
4. $7\sqrt{5} - \sqrt{5}$
5. $2\sqrt{2} + \sqrt{8}$
6. $\sqrt{8} - \sqrt{2}$
7. $2\sqrt{6} - \sqrt{24}$
8. $2\sqrt{a} + \sqrt{a}$
9. $2\sqrt{a} - \sqrt{a}$
10. $\sqrt{16} + \sqrt{2}$

Problem Set 12-6

1. Simplify. Indicate the restrictions on the variable.

   (a) $5\sqrt{5} + 2\sqrt{5}$

   (b) $5\sqrt{5} - 2\sqrt{5}$

   (c) $8\sqrt{a} + 2\sqrt{a}$

   (d) $8\sqrt{a} - 2\sqrt{a}$

   (e) $3\sqrt{2} + \sqrt{2}$

   (f) $3\sqrt{2} - \sqrt{2}$

2. Simplify.

   (a) $\sqrt{2} + \sqrt{8}$

   (b) $\sqrt{2} - \sqrt{8}$

   (c) $3\sqrt{3} + \sqrt{18}$

   (d) $3\sqrt{3} - \sqrt{18}$

   (e) $2\sqrt{12} + 3\sqrt{2}$

   (f) $\sqrt{75} - 3\sqrt{2}$

3. Simplify. Indicate the restrictions on the variable.

   (a) $\sqrt{75} + 2\sqrt{8}$

   (b) $\sqrt{98} - \sqrt{32}$

   (c) $\sqrt{8a} + 2\sqrt{a}$

   (d) $\frac{1}{2}\sqrt{32} + \sqrt{200}$

   (e) $\sqrt{\frac{1}{2}} + 2\sqrt{2}$

   (f) $\sqrt{\frac{2}{3}} - \frac{1}{\sqrt{6}}$

4. Simplify.

   (a) $\sqrt{8} + \sqrt{32} - \sqrt{2}$

   (b) $8\cdot\sqrt{\frac{1}{2}} - \frac{1}{2}\cdot\sqrt{8}$

   (c) $2\sqrt{18} + \frac{1}{2}\sqrt{16} - \sqrt{20}$

   (d) $\sqrt{\frac{5}{9}} + \sqrt{\frac{9}{5}}$

   (e) $2\sqrt{\frac{5}{3}} + \frac{1}{2}\sqrt{60} + \sqrt{5^3 \cdot 3}$

   (f) $\frac{1}{4}\sqrt{288} - \frac{1}{6}\sqrt{72}$

Problem Set 12-6
(continued)

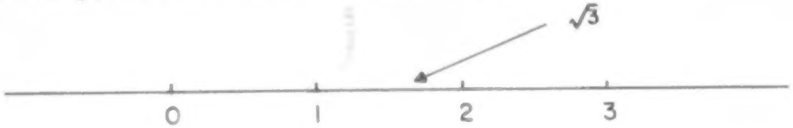
5. Simplify. Indicate the restrictions on the variable.
   (a) $|a| \sqrt{16a} + 2\sqrt{a^3}$
   (b) $\sqrt{18x^2} + \sqrt{2x^2}$
   (c) $5\sqrt{3t^2} - \sqrt{27t^2}$
   (d) $\frac{8}{3}\sqrt{\frac{5}{4}b} + 5\sqrt{\frac{5b}{16}} + 3\sqrt{\frac{45}{16}b}$

12-7. Approximate Square Roots in Decimals.

We have said that the simplest form in which to write the square root of 3 is

$$\sqrt{3}\ .$$

This is a real number. Therefore it must be associated with some point on the number line. Since we know that $1^2 = 1$, and $2^2 = 4$, our point must lie somewhere between 1 and 2.



But the symbol $\sqrt{3}$ doesn't really tell us very much about just where this is. Therefore we would like to be able to express this number in another way.

Since we know that $\sqrt{3}$ is not a rational number, we cannot write it as a fraction with integers for numerator and denominator. We also cannot write it exactly as a decimal with a fixed number of decimal places, since such decimals are names for rational numbers (for example, $.4 = \frac{2}{5}$, $.301 = \frac{301}{1000}$)

On the other hand we can find a decimal expression which will be close to the real square root we are looking for. This expression will tell us more about where the point should lie.

Let's begin by squaring some numbers between 1 and 2, written as numerals with one decimal place. By actual multiplication we see that

$$(1.5)^2 = 2.25, \quad (1.6)^2 = 2.56, \quad (1.7)^2 = 2.89,$$
$$\text{and} \quad (1.8)^2 = 3.24.$$

This should make it clear that the point associated with $\sqrt{3}$ lies somewhere between 1.7 and 1.8.

$\sqrt{3}$
0
1
1.7
1.8
2

We can express this idea using inequality symbols as follows:

$$1.7 < \sqrt{3} < 1.8.$$

We can read this in two ways. We say that $\sqrt{3}$ is greater than 1.7 and is less than 1.8. Another way is to say that

$\sqrt{3}$ is between 1.7 and 1.8.

The method which we have just been using consists of two steps. We first determine the two integers between which our square root lies. This locates our point somewhere in an interval which is one unit long.

In the next step we divide the interval into tenths and then locate our point somewhere in one of these smaller intervals.

Check Your Reading

1. Is there an exact decimal name for $\sqrt{3}$? Why or why not?
2. What is the meaning of the statement "$1.7 < \sqrt{3} < 1.8$"?

Oral Exercises 12-7a

Between what two integers do each of the following lie?

1. $\sqrt{7}$
2. $\sqrt{13}$
3. $\sqrt{33}$
4. $\sqrt{75}$
5. $\sqrt{107}$
6. $\sqrt{53}$
7. $\sqrt{99}$
8. $\sqrt{82}$
9. $\sqrt{140}$
10. $\sqrt{27}$

Problem Set 12-7a

Locate each of the following in an interval of one-tenth as in the example in the text.

1. $\sqrt{26}$
2. $\sqrt{65}$
3. $\sqrt{123}$
4. $\sqrt{85}$
5. $\sqrt{13}$
6. $\sqrt{21}$
7. $\sqrt{55}$
8. $\sqrt{90}$
9. $\sqrt{63}$
10. $\sqrt{19}$

We have learned that $\sqrt{3}$ lies between 1.7 and 1.8. By actual multiplication we see that

$$(1.70)^2 = 2.8900, \quad (1.71)^2 = 2.9241, \quad (1.72)^2 = 2.9584,$$
$$(1.73)^2 = 2.9929, \qquad (1.74)^2 = 3.0276 .$$

From this we can locate $\sqrt{3}$ in an even smaller interval. We can say that

$1.73 < \sqrt{3} < 1.74$. Can you see why?

We call such numbers as 1.73 or 1.74 approximations to $\sqrt{3}$. Which is closer?

These two numbers have been determined by squaring every two-place decimal from 1.70 to 1.74. But this approach can mean lots of work and some wasted time.

There are better methods for finding an approximate square root. We will show one of these methods by means of examples.

Consider the problem of finding an approximation to $\sqrt{10}$. As in the previous examples, we first locate the two integers between which our square root must lie. Can you see what these two integers are? Suppose we try 3 and 4. $3^2 = 9$, and $4^2 = 16$.

In this method we want to pick the integer which we think is closer to $\sqrt{10}$. What is your choice? No doubt it is 3. We call 3 in this problem a

first approximation

to $\sqrt{10}$. It is convenient, now, to make use of a new symbol " ≈ " meaning, "is approximately equal to." Thus we write

$$3 \approx \sqrt{10}\ .$$

Our next job is to find a rational number which is closer to $\sqrt{10}$ than 3. We will call this new number a second approximation. Now suppose we did have a number p which is the exact square root of 10. That is, let $p = \sqrt{10}$. We could then say that

$$p \cdot p = 10.$$

But we know that 3 is not exactly equal to p. In fact, since $3 \cdot 3 = 9$, we know that 3 is smaller than p. Do you see why?

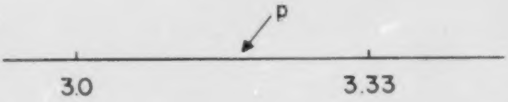
Let's now form the sentence

$$3 \cdot n = 10.$$

Since 3 is smaller than p, then, if this is a true sentence, n must represent a number larger than p. Multiplying both sides by $\frac{1}{3}$ we obtain

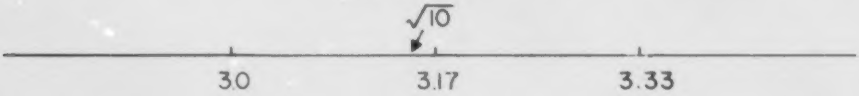
$n = \frac{10}{3}$ which can be written approximately as 3.33.

We have shown that p lies between 3 and 3.33.



We take as our second approximation the point half-way between 3 and 3.33. Another way of saying this is that we take the average of 3 and 3.33 which gives us

$$\frac{3 + 3.33}{2} = 3.17.$$



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Thus our second approximation is 3.17.
It is interesting to note that

$$(3.17)^2 = 10.0489.$$

This tells us that 3.17 is closer to $\sqrt{10}$ than 3 is.

This process could now be done again. We could take 3.17 as a new approximation. As before, we would divide this into 10. We would then take 3.17 and $\frac{10}{3.17}$, find the average of these two, that is, we would calculate

$$\frac{3.17 + \frac{10}{3.17}}{2},$$

and find an even closer approximation.

In general, however, we will perform the operation only once. To review the steps, let's consider another example. Find an approximate value of $\sqrt{22}$.

We see that $4^2 = 16$ and $5^2 = 25$. Which one is closer? If we think that 5 is closer, what is the next step? We say that $5 \cdot n = 22$, so $n = \frac{22}{5}$, which is 4.4.

As before, we see that $\sqrt{22}$ lies between 4.4 and 5. Therefore we find the average of 4.4 and 5, which is $\frac{4.4 + 5}{2} = 4.7$; we can now say that as a second approximation

$$\sqrt{22} \approx 4.7$$

Check by multiplication that 4.7 is a closer approximation than 5.

Check Your Reading

1. In the text, what was the first approximation to $\sqrt{10}$?
2. How is the sentence "$\sqrt{10} \approx 3$" read?
3. If 3 is used as the first approximation to $\sqrt{10}$, how is the second approximation determined?

Oral Exercises 12-[illegible]

1. Give a first approximation for each of the following:

   (a) $\sqrt{12}$

   (b) $\sqrt{26}$

   (c) $\sqrt{80}$

   (d) $\sqrt{66}$

   (e) $\sqrt{7}$

   (f) $\sqrt{120}$

   (g) $\sqrt{99}$

   (h) $\sqrt{75}$

   (i) $\sqrt{42}$

   (j) $\sqrt{5}$

2. Give a second approximation for each of the following:

   (a) $\sqrt{20}$

   (b) $\sqrt{30}$

   (c) $\sqrt{72}$

   (d) $\sqrt{56}$

   (e) $\sqrt{5}$

   (f) $\sqrt{82}$

Problem Set 12-7b

1. Find first approximations for the following numbers.

   (a) $\sqrt{17}$

   (b) $\sqrt{41}$

   (c) $\sqrt{83}$

   (d) $\sqrt{130}$

   (e) $\sqrt{107}$

   (f) $\sqrt{11}$

2. Find second approximations for the following numbers.

   (a) $\sqrt{17}$

   (b) $\sqrt{41}$

   (c) $\sqrt{83}$

   (d) $\sqrt{130}$

   (e) $\sqrt{107}$

   (f) $\sqrt{11}$

3. Find second approximations for the following numbers.

   (a) $\sqrt{23}$

   (b) $\sqrt{19}$

   (c) $\sqrt{37}$

   (d) $\sqrt{89}$

   (e) $\sqrt{113}$

   (f) $\sqrt{48}$

Problem Set 12-7b
(continued)

4. Find decimal approximations for the following, given $\sqrt{2} \approx 1.414$ and $\sqrt{3} \approx 1.732$.

   Example: $\sqrt{50}$

   Solution:
   $$\begin{aligned} \sqrt{50} &= \sqrt{5^2 \cdot 2} \\ &= 5\sqrt{2} \\ \sqrt{50} &\approx 5\,(1.414) \\ \sqrt{50} &\approx 7.070 \end{aligned}$$

   (a) $\sqrt{72}$

   (b) $\sqrt{18}$

   (c) $\sqrt{108}$

   (d) $\sqrt{48}$

   (e) $\frac{1}{\sqrt{3}}$ Hint: rationalize the denominator.

   (f) $\sqrt{2450}$

5. The diagonal of a rectangle can be found by the formula
   $$d = \sqrt{\ell^2 + w^2}$$
   where d is the number of units in the length of the diagonal, ℓ is the number of units in the length, and w is the number of units in the width, all measured in the same unit. Find a second approximation to the diagonal of a rectangle in each of the following cases.

   (a) ℓ is 5 feet and w is 4 feet.
   (b) ℓ is 5 inches and w is 3 inches.
   (c) ℓ is 7 yards and w is 5 yards.

12-8. Cube Roots and n^{th} Roots.

We have defined square roots of real numbers as follows. We say that b is a square root of a if $b^2 = a$.

Consider, now, the number 8. We see that 8 can be written as

$$2 \times 2 \times 2$$

which is the same as

$$2^3.$$

Likewise

$$27 = 3 \times 3 \times 3 \qquad \text{and} \qquad 64 = 4 \times 4 \times 4$$
$$= 3^3 \qquad\qquad\qquad\qquad = 4^3$$

We call 2 the "cube root" of 8. Similarly 3 is the cube root of 27, and 4 is the cube root of 64. Can you see what the cube root of 125 might be?

In general we say that a real number b is the cube root of a real number a, if

$$b^3 = a.$$

The study of cube roots brings in some ideas which differ somewhat from those involved in square roots. You recall, for example, that a square root of 49 is 7; another square root of 49 is -7.

However, in the case of the cube root of 8, we have shown that 2 is a cube root. But what about -2? What is the value of $(-2)(-2)(-2)$?

Do you see that it is -8? Therefore we say that 2 is the only real number which is the cube root of 8. Likewise 3 is the only real number which is the cube root of 27.

Since we know that $(-2)^3 = -8$, it follows that -2 is the cube root of -8. Thus we can say that there are real numbers which are cube roots of negative real numbers. In fact it can be shown that every real number has exactly one real number as its cube root.

Again we make use of the radical sign to indicate cube roots. In this case we place a "3" in the symbol in this way: $\sqrt[3]{}$ Thus,

$$\sqrt[3]{125} = 5 \qquad\qquad \sqrt[3]{-64} = -4.$$

The "3" in the symbol is called the index of the radical. The cube root radical can represent a positive or a negative real number; since there is no confusion as to which one is meant. We also say that

$$\sqrt[3]{0} = 0.$$

Check Your Reading

1. What is the cube root of 8?
2. Give another name for $\sqrt[3]{27}$.
3. If $b = \sqrt[3]{a}$, then $b^3 =$ _____.
4. Give another name for $\sqrt[3]{0}$.
5. If x is a positive number, is $\sqrt[3]{x}$ positive or negative?
6. If x is a negative number, is $\sqrt[3]{x}$ positive or negative?

Oral Exercises 12-8a

Give a simpler name for each of the following:

1. $\sqrt[3]{27}$
2. $\sqrt[3]{-8}$
3. $\sqrt[3]{64}$
4. $\sqrt[3]{\frac{1}{8}}$
5. $\sqrt[3]{-\frac{1}{27}}$
6. $\sqrt[3]{a^3}$ Is there any restriction on the domain of a?
7. $\sqrt[3]{2^3 \quad 3}$
8. $\sqrt[3]{\frac{1}{3^3 \cdot 5^3}}$
9. $\sqrt[3]{216a^3}$
10. $-\sqrt[3]{-8}$

Problem Set 12-8a

1. Give a simpler name for each of the following:
   (a) $\sqrt[3]{-27}$
   (b) $-\sqrt[3]{8}$
   (c) $\sqrt[3]{\frac{8}{64}}$
   (d) $\sqrt[3]{-\frac{64}{27}}$
   (e) $\sqrt[3]{125}$
   (f) $\sqrt[3]{3^3 \cdot 7^3}$

2. Give a simplier name for each of the following:
   (a) $-\sqrt[3]{-27}$
   (b) $-\sqrt[3]{a^3}$
   (c) $\sqrt[3]{(-b)^3}$
   (d) $\sqrt[3]{216x^3}$
   (e) $\sqrt[3]{64x^3y^3}$
   (f) $\sqrt[3]{\frac{m^3}{125}}$

Problem Set 12-8a
(continued)

3. Find the truth sets of:

(a) $x^3 = -27$ (d) $\sqrt[3]{x} = \frac{1}{2}$

(b) $4m^3 = 32$ (e) $5m^3 - 135 = 0$

(c) $\frac{1}{2}t^3 - 12 = 20$ (f) $3\sqrt[3]{m} = 243$

In dealing with square roots we learned a very important theorem. This told us that if the square root of an integer is a rational number, then it must also be an integer. Thus we were able to see that numbers like $\sqrt{5}$ and $\sqrt{23}$ are not rational numbers.

The same type of argument holds for cube roots. We can show that if the cube root of an integer is a rational number, then it must be an integer. Thus we have cube roots which are irrational numbers. Take for example

$$\sqrt[3]{20}\ .$$

The proper prime factors of 20 are $2 \times 2 \times 5$. There is no integer which is the cube root of 20, and therefore no rational number. But the cube root of 20 is associated with a point on the real number line. It is a real number.

In general, the idea of root goes beyond square and cube root. Consider a number such as 81. Its prime factors are

$$3 \times 3 \times 3 \times 3.$$

In this case we say that 3 is a fourth root of 81, since $81 = 3^4$. Since $(-3)^4 = 81$, -3 is also a fourth root of 81.

If we have a number such that $a^5 = b$, then we say that a is a fifth root of b. We can summarize by saying that if

$$x^n = y,$$

where x and y are real numbers and n is a positive integer, then

$$x \text{ is an "}n^{\text{th}}\text{" root of } y.$$

As in the case of cube roots and square roots, we can find n^{th} roots which are irrational, where n is any positive integer. In addition to n^{th} roots of real numbers there are many other real numbers associated with points on the number line which are irrational. You have heard of such numbers: π is one of these. In fact it can be proved that there are more numbers which are neither rational nor the n^{th} roots of rational numbers than there are of any other type.

Check Your Reading

1. Why isn't there a rational cube root of 20?
2. What does "x is an n^{th} root of y" mean?

Problem Set 12-8b

1. Which of the following symbols represent real numbers and which do not?

$$\sqrt{4},\quad \sqrt[3]{8},\quad \sqrt{-16},\quad \sqrt[3]{-8},\quad \sqrt{2},\quad -\sqrt{3},$$

$$-\sqrt{9},\quad -\sqrt[3]{-64},\quad \sqrt[4]{2^4},\quad \sqrt[4]{(-2)^5},\quad \sqrt{2\tfrac{1}{4}}$$

2. Classify the real numbers named in Problem 1 into the "smallest" of the following sets to which each belongs: the integers, the rationals, the irrationals.

Summary

1. For a positive real number a, the symbol $\sqrt{a}$ indicates the positive square root of a.
2. In general we say that $\sqrt{a} = b$, if b is positive and $b^2 = a$.
3. $\sqrt{0} = 0$.
4. If any integer has a square root which is a rational number, then this square root must itself be an integer.

5. A real number which is not rational is called an irrational number. Numbers such as $\sqrt{2}$, $\sqrt{3}$, $\sqrt{10}$, are irrational.

6. For any two non-negative real numbers a and b, $(\sqrt{a})(\sqrt{b}) = \sqrt{ab}$.

7. If a is a non-negative real number, and if b is a positive real number, then $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$.

8. The following examples illustrate what we mean by simplifying a radical.

(a) $\sqrt{32a^5} = \sqrt{(16a^4)(2a)}$

$= 4a^2\sqrt{2a}\,,\quad a \geq 0$

(b) $\sqrt{50x^2} = 5|x|\sqrt{2}$

(c) $\sqrt{\frac{3}{7}} = \sqrt{\frac{3}{7}} \cdot \sqrt{\frac{7}{7}} = \frac{\sqrt{21}}{7}$

$\sqrt{\frac{3}{7}} = \sqrt{\frac{3}{7}} \cdot \sqrt{\frac{3}{3}} = \frac{3}{\sqrt{21}}$

$\frac{\sqrt{21}}{7}$ and $\frac{3}{\sqrt{21}}$ are each considered to be a simpler form than $\sqrt{\frac{3}{7}}$.

The process used to arrive at $\frac{\sqrt{21}}{7}$ is called rationalizing the denominator. The form $\frac{3}{\sqrt{21}}$ was obtained by rationalizing the numerator.

9. For any real number a, $\sqrt[3]{a} = b$ if $b^3 = a$.

10. $\sqrt[3]{a}$ is called the cube root of a. The number 3 is called the index of the radical.

11. Every real number has exactly one real cube root. The cube root of a negative number is negative. The cube root of a positive number is positive. $\sqrt[3]{0} = 0$.

12. In contrast to this every positive real number has two real square roots which are the opposite of each other.

13. A real number b is called an n^{th} root of a real number a if $b^n = a$, where n is a positive integer.

14. In addition to irrational n^{th} roots there are many more real numbers which are irrational. This latter type of irrational number makes up the largest of all the sets of real numbers.

Summary of the Fundamental Properties of Real Numbers.

Suppose you were asked the following question: "Just what do we mean when we say, 'The Real Numbers'?" This is a difficult one to answer. We might say something like, "These are the numbers people use every day," or, "They are the things we count and measure with".

Actually neither of these statements would tell very much. As a matter of fact much of the work we have been doing so far has consisted of an attempt to arrive at an understanding of what the real number system is. In doing this we haven't said very much about what the real numbers themselves are. On the other hand we have learned a great deal about how they behave. We have studied operations, relations, and properties of these. But it is just these ideas which give us an understanding of real numbers. Thus, the best way to deal with our question might be to discuss these properties.

It turns out that a full definition of the real number system can be given by means of certain of the properties we have studied plus one additional property. We will not discuss this last property in detail since it involves complications beyond the range of this course. However, it is interesting to note that if we include all but this last property, we have a full definition of the rational numbers.

It is this last property which, in a sense, brings in the irrational numbers; that is, it fills up the number line. For this reason we can refer to it as the completeness property. Since our definitions of the real number system must include this, we will mention it by name, but not give a description.

The following, then, defines the real number system. It should also provide a very helpful review.

The real number system is a set of elements called real numbers. The system has two operations: addition, with the symbol "+", and multiplication with the symbol "·". The system also has an order relation "is less than" with the symbol "<". The two operations and the relation have the following properties.

1. For any real numbers a and b, $a + b$ is a real number.
2. For any real numbers a and b, $a + b = b + a$. (commutative property of addition)
3. For any real numbers a, b, and c, $(a + b) + c = a + (b + c)$. (associative property of addition)
4. There is a special real number 0 such that, for any real number a, $a + 0 = a$. (identity element of addition)
5. For every real number a there is a real number $-a$ such that $a + (-a) = 0$. (inverse elements of addition)
6. For any real numbers a and b, $a \cdot b$ is a real number.
7. For any real numbers a and b, $a \cdot b = b \cdot a$. (commutative property of multiplication)
8. For any real numbers a, b, and c, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$. (associative property of multiplication)
9. There is a special real number 1 such that, for any real number a, $a \cdot 1 = a$. (identity element of multiplication)

10. For every real number a different from 0, there is a real number $\frac{1}{a}$ such that $a \cdot (\frac{1}{a}) = 1$. (inverse elements of multiplication)

11. For any real numbers $a, b,$ and c, $a(b + c) = a \cdot b + a \cdot c$. (distributive property)

12. For any real numbers a and b, exactly one of the following is true: $a < b$, $a = b$, $b < a$. (comparison property of order)

13. For any real numbers $a, b,$ and c, if $a < b$ and $b < c$, then $a < c$. (transitive property of order)

14. For any real numbers $a, b,$ and c, if $a < b$ then $a + c < b + c$. (addition property of order)

15. For any real numbers $a, b,$ and c, if $a < b$ and c is positive, then $c \cdot a < c \cdot b$; if $a < b$ and c is negative, then $c \cdot b < c \cdot a$. (multiplication property of order)

16. The Completeness Property.

Some of the properties which we have studied are not included in this list. We have left these out for a definite reason. It is because these other properties can be proved by using the ones which appear in the list. We shall include below a set of additional properties. We hope that you will try to discover how these can be shown to be true for real numbers.

Since the first set of properties is all that we need in order to prove, or discover, the others, this first set, as we said before, completely defines The Real Number System.

Additional Properties

17. Any real number x has just one additive inverse, namely $-x$.

18. For any real numbers a and b,

$$-(a + b) = (-a) + (-b).$$

19. For real numbers a, b, and c, if $a + c = b + c$, then $a = b$.

20. For any real number a, $a \cdot 0 = 0$.

21. For any real number a, $(-1)a = -a$.

22. For any real numbers a and b, $(-a)b = -(ab)$ and $(-a)(-b) = ab$.

23. Any real number x different from 0 has just one multiplicative inverse, namely $\frac{1}{x}$.

24. The number 0 has no reciprocal.

25. The reciprocal of a positive number is positive, and the reciprocal of a negative number is negative.

26. The reciprocal of the reciprocal of a non-zero real number a, is a.

27. For any non-zero real numbers a and b,

$$\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{ab}.$$

28. For real numbers a and b, if $ab = 0$ then $a = 0$ or $b = 0$.

29. For real numbers a, b, and c with $c \neq 0$, if $ac = bc$, then $a = b$.

30. For any real numbers a and b, if $a < b$, then $-b < -a$.

31. If a and b are real numbers such that $a < b$, then there is a positive number c such that $b = a + c$.

32. If a and b are positive real numbers, and if $a < b$, then $\frac{1}{b} < \frac{1}{a}$.

As an illustration of how a proof might be carried out, let us consider Number 19 from the list of additional properties. "For real numbers a, b, and c, if $a + c = b + c$, then $a = b$." We assume that a, b, and c, are given and that

$$a + c = b + c.$$

We can now use fundamental property Number 5, which, with a change of letter can be stated,

> For every real number c there is a real number $-c$ such that $c + (-c) = 0$.

Thus, we can now say that

$$(a + c) + (-c) = (b + c) + (-c)$$

since the right and left sides of the equations name the same number.

Also $(a + c) + (-c) = a + \big(c + (-c)\big)$,

and $(b + c) + (-c) = b + \big(c + (-c)\big)$. by property Number 3

Hence $a + 0 = b + 0$.

Now by property Number 4 we get

$$a = b,$$

which is the statement we were trying to prove.

Review Problem Set

1. Tell which of the following symbols represent real numbers and which do not. Further identify each number as an element of the "smallest" set of numbers to which it belongs. (Integers, rational numbers, or irrational numbers).

$$-\sqrt{2},\quad \sqrt{9},\quad \sqrt{-7},\quad (\sqrt{3})^2,\quad \frac{1}{\sqrt{2}},\quad \frac{\sqrt{2}}{3-3},\quad 3+2,$$

$$\frac{2}{\sqrt{-2}},\quad -\sqrt{36},\quad \frac{5}{4},\quad \sqrt{\frac{2}{3}},\quad \frac{6}{0},\quad \frac{63}{7},\quad \frac{0}{6}$$

Review Problem Set
(continued)

2. Simplify:

(a) $\sqrt{2}\sqrt{6}$ (d) $\sqrt{\frac{9}{7}}$

(b) $\frac{\sqrt{18}}{\sqrt{6}}$ (e) $\sqrt{1024}$

(c) $\sqrt{2}(\sqrt{18}+\sqrt{2})$ (f) $\sqrt{1078}$

3. Find the truth sets of:

(a) $x^2 = -3$ (d) $m\sqrt{3} = \sqrt{108}$

(b) $4y^2 = 20$ (e) $5t^2 + 9 = 19$

(c) $\sqrt{5} + z = \sqrt{20}$ (f) $\sqrt{m^2} = m$

4. Which of the following sentences are true and which are false?

(a) $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} = \frac{\sqrt{3}+\sqrt{2}}{\sqrt{6}}$

(b) $\sqrt{2} + \sqrt{3} = \sqrt{5}$

(c) $\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

(d) For every x, $\sqrt{x^2} = |x|$

(e) $(\sqrt{2}+\sqrt{3})^2 = 5$

(f) $16 < \sqrt{260} < 17$

5. Simplify indicating the domain of the variable when restricted.

(a) $\sqrt[3]{8a^3}$ (d) $\sqrt[3]{27b^3}$

(b) $\sqrt{18x^2}$ (e) $-\sqrt{32x^3y^2}$

(c) $\sqrt{8a}\sqrt{2a}$ (f) $\frac{\sqrt{20a}}{\sqrt{5a}}$

Review Problem Set
(continued)

6. Simplify:

(a) $\sqrt{\frac{2}{3}}\sqrt{\frac{5}{6}}\ \sqrt{\frac{3}{2}}$

(b) $(\sqrt{3} - \sqrt{2})(\sqrt{3} + \sqrt{2})$

(c) $\sqrt{\frac{1}{2}} - \sqrt{\frac{9}{8}}$

(d) $\sqrt{48} - \sqrt{75} + \sqrt{12}$

(e) $\sqrt{\frac{16}{3}} + \frac{1}{\sqrt{3}} - \sqrt{27}$

(f) $\sqrt{6}\sqrt{24} - \sqrt{2}\sqrt{72}$

7. Multiply

(a) $(x - 3)(x + 2)$

(b) $(y + 3)(y + 1)$

(c) $(m + 5)(m - 6)$

(d) $(2n - 1)(n + 1)$

(e) $(a - b)(a + b)$

(f) $(2x - 1)(2x + 1)$

8. Simplify. Indicate the restrictions on the variable where necessary.

(a) $\dfrac{\frac{1}{3}}{\frac{2}{3}}$

(b) $\dfrac{\frac{1}{2}}{\frac{5}{6}}$

(c) $\dfrac{\frac{3}{4}}{\frac{4}{3}}$

(d) $\dfrac{\frac{7m^2}{4q}}{\frac{-2m}{8}}$

(e) $\dfrac{\frac{1}{3a}}{\frac{2}{5a}}$

(f) $\dfrac{1 + \frac{1}{b}}{\frac{1}{b}}$

9. A boy has $\$\,1.75$ in change. If he has twice as many dimes as nickels and $\frac{1}{5}$ as many quarters as dimes, how many of each coin does he have?

10. A plane has enough gasoline for 6 hours of flight. How many miles can it fly away from its base before it must turn back, if its rate is 100 mph away and 120 mph returning?

Chapter 13

POLYNOMIALS

13-1. Polynomials.

In a previous chapter we were interested in learning how to express numbers in factored form. We now wish to find out whether it is possible to write more complicated expressions in factored form. In other words, if we are given a certain type of phrase, can we write it as the indicated product of simpler phrases?

A partial answer to this question was already given when we made use of the distributive property. For example, we have learned that

$$\begin{aligned} 5a(a + b) &= 5a \cdot a + 5a \cdot b \\ &= 5a^2 + 5ab. \end{aligned}$$

In this case we say that $5a(a + b)$ is the factored form of $5a^2 + 5ab$. By a similar application of the distributive property we see that

$$x^2 + 3x \text{ can be written as } x(x + 3)$$

$$\begin{aligned} \text{since } x^2 + 3x &= x \cdot x + 3 \cdot x \\ &= x \cdot x + x \cdot 3 \\ &= x(x + 3). \end{aligned}$$

By the same property

$$\begin{aligned} 3y^2 + 6y &= 3 \cdot y \cdot y + 3 \cdot 2 \cdot y \\ &= 3 \cdot y \cdot y + 3 \cdot y \cdot 2 \\ &= (3y)y + (3y)2 \\ &= 3y(y + 2). \end{aligned}$$

In the above examples we say that

$x(x + 3)$ is the factored form of $x^2 + 3x$,

and $3y(y + 2)$ is the factored form of $3y^2 + 6y$.

In working the above examples we also used some other properties. Can you see what they were?

Oral Exercises 13-1a

Express each of the following in the factored form:

1. (a) $m^2 + 2m$
   (b) $3n^2 + 6$
   (c) $3y^2 + 6y$
   (d) $a^2 - a$
   (e) $b^2 + b$
   (f) $2b^2 + 4$
   (g) $2x^2 + 6x$
   (h) $4z^2 - 10$
   (i) $4r^2 - 10r$

In our study of the factored forms of numbers in Chapter 11 we worked with positive integers. We wrote positive integers as products of other numbers which were themselves positive integers. The point to be noted here is that the individual factors were in all cases numbers of the same kind as the number we started with.

For example, we might have said the following:

$$6 = (-2)(-3)$$

This is a true sentence. However, the factors on the right are negative integers. The number on the left is a positive integer. Since we wanted the factors to be the same kind of numbers as the original number, we said that the factored form should be

$$(2)(3).$$

Another way of factoring the number 6 might have been to write

$$6 = (\tfrac{1}{2})(12).$$

Here the number $\frac{1}{2}$ is a positive number; but it is not an integer. Therefore it is not the same kind of number as the number 6, and thus we did not include $\frac{1}{2}$ as a factor of 6.

In this chapter we shall be working with phrases rather than with individual numbers. The important thing to keep in mind is that when we wish to write a certain phrase in factored form we will want the factors themselves to be expressions of the same kind as the phrase we started with. It is necessary, then, that we know what we mean by phrases of the same kind, or phrases of a special type.

When we refer to a number as a positive integer, we know what we are talking about. In a list of numbers such as

$$3 \qquad \frac{3}{4} \qquad (-6) \qquad \frac{1}{2} \qquad (-\frac{1}{2}) \qquad -25 \qquad 7$$

the first and last are positive integers. The others are not. But suppose we consider phrases such as

$$\frac{x^2 + 5}{2x - 1}, \qquad \frac{4}{x + 3}, \qquad x^2 + 3x, \qquad \frac{1}{x}, \qquad x^3 + x - 7.$$

The problem of selecting certain ones is not so easy. We would probably agree that they are not all of the same type. Which ones are different from the others? We will be interested in one special type. The kind we are interested in are the third and last expressions.

If you were asked to indicate the difference between these two and the others, how would you answer the question? All 5 expressions contain a variable. Others beside the third and last show exponents, which implies multiplication. They also show addition and subtraction. Do you see, however, that the third and last are the only ones which have no indicated division?

The expressions

$$x^2 + 3x \qquad \text{and} \qquad x^3 + x - 7$$

are examples of what we call polynomials. The other expressions are not polynomials.

To help in our understanding let us examine the way in which a certain type of polynomial is formed. As a start, consider the set consisting of all the integers, that is the set

$$I = \{\ldots, (-3), (-2), (-1), 0, 1, 2, 3, \ldots\}.$$

Then let us consider one, two, or more variables, depending on whether we want to form polynomials in one variable, two variables, or more. We shall assume, as before, that the domain of these variables is the set of all real numbers.

We are now ready to describe what we mean by a polynomial. In the first place a polynomial may consist of any numeral for an element in our set or a variable itself. Thus, "x" is a polynomial, "-x" is a polynomial, "5" is a polynomial, "(-7)" is a polynomial.

A polynomial is also

> any expression which indicates addition, subtraction, multiplication, or taking opposites of any elements of our set and the variables.

For example, let us begin with a variable x and indicate multiplication of x by x. This gives us the expression x^2. Then indicate multiplication of 5 by x. This gives us $5x$. Then indicate the sum of these expressions. This gives us $x^2 + 5x$. Next indicate subtraction of 7. Our resulting expression is

$$x^2 + 5x - 7$$

which is a polynomial. Note that the expressions x, x^2, $5x$, $x^2 + 5x$ are also polynomials.

The indicated operations of addition, subtraction, multiplication, or taking opposites could have continued much longer, giving us something like

$$x^3 + 2x^2 + 10x - 14$$

or even more complicated expressions. These would still be polynomials as long as we used no other than the four operations listed above a finite number of times. However, as we said before, expressions like

$$\frac{x+5}{2x-1} \qquad \frac{4}{x+3} \qquad \frac{1}{x} \qquad \frac{3}{x^2}$$

are not polynomials. Why not? Do you see that they all indicate the operation of division? Further examples of polynomials are

$$x^3$$

$$x^2 - 6x + 12$$

$$7x - 5$$

The original set of numbers which we started with in forming our polynomials was the set of integers. There are other types of polynomials for which the original set of numbers may be a different set, such as the rational numbers or the real numbers. For this reason it is important to distinguish the different types as polynomials over the rationals, polynomials over the reals, polynomials over the integers. However, we shall for the most part be concerned with the last type. For convenience, therefore, we shall merely use the word polynomials in the early sections of this chapter to mean polynomials over the integers. Later, when we are working with other number sets, we shall specify the type of polynomial being considered.

Our definition of polynomials applies to polynomials in any number of variables. Examples of polynomials in two variables are

$$5x + x^2y - y^3 + 6$$

$$5st - s^2 + 3t$$

$$a^2 - 4ab + b^2.$$

Examples of polynomials in three variables are

$$2xyz + z - y + 1$$

$$3a + 2b - 4c$$

$$r^2 + rt^2 - 5st + 7$$

Check Your Reading

1. Why do we consider $(2)(3)$ a better factored form of 6 than $(-2)(-3)$? $(\frac{1}{2})(12)$?

Check Your Reading
(continued)

2. Which of the following are positive integers?

$$3, \ \frac{3}{4}, \ (-6), \ \frac{1}{2}, \ (-\frac{1}{2}), \ (-25), \ 7$$

3. How does the phrase $\frac{x^2 + 5}{2x - 1}$ differ from the phrase $x^2 + 5x$?

4. Give an example of a polynomial in one variable.

5. Which of the following are polynomials?

$$\frac{x^2 + 5}{2x - 1}, \ \frac{4}{x + 3}, \ x^2 - 6x + 12, \ r^2 + rt^2 - s + 7, \ \frac{1}{x}.$$

6. Polynomials such as $5x + x^2y - y^3 + 6$ and $a^2 - 4ab + b^2$ are polynomials in how many variables?

7. What about the polynomial $2xyz + z - y + 1$?

8. Is it possible to write a polynomial in more than three variables?

9. What operations may be indicated in a polynomial?

Oral Exercises 13-1b

1. Tell which of the following are polynomials.

   (a) $2a$

   (b) 3

   (c) $\frac{1}{2x^2 + 6x}$

   (d) $2ay$

   (e) $x^2 + 6x$

   (f) $\frac{1}{x^2}$

   (g) $\frac{3}{x}$

   (h) $-4a$

   (i) $-6ab^2 + 12ab + 18a$

   (j) $\frac{x + 1}{x + 4}$

   (k) $\frac{a}{a^2 + 4a + 4}$

2. In the exercise above indicate which polynomials are polynomials in one variable; in two variables; in three variables.

Problem Set 13-1b

1. Write three polynomials in one variable.

2. Write three polynomials in two variables.

3. Write three polynomials in three variables.

4.
   (a) Write the variable y.
   (b) Indicate the sum of 3 and y.
   (c) Indicate the product of this sum and 2.
   (d) Indicate that 9 is subtracted from this product.
   (e) Is the resulting expression a polynomial?

5.
   (a) Write any letter as a variable and take its opposite.
   (b) Indicate the product of 4 and the variable.
   (c) Indicate the sum of 6 and the product.
   (d) Indicate the product of 3 and the sum.
   (e) Indicate the quotient of the result in (d) by the opposite of the variable you chose in (a).
   (f) Does the resulting expression involve only the operations permitted by the definition of a polynomial? Is it a polynomial?

6. Perform the indicated operations and simplify. Is the result a polynomial over the integers?
   (a) $2x(x - 2)$
   (b) $y(3y + 5)$
   (c) $(m + 2)(m + 3)$
   (d) $\frac{1}{2}(t + 1)$
   (e) $(u + 6)(u - 2)$
   (f) $xy(xy^2 + x^2y)$

7. Write each of the following polynomials in factored form where the factors are of the same kind as the polynomial.
   (a) $6x + 3$
   (b) $5y + 5z$
   (c) $2h^2 + h$
   (d) $4z^2 - 8z$
   (e) $3x^2 - 2x$
   (f) $ax^2 + ax$

13-2. Factoring.

In the beginning of this chapter we asked the question, "If we are given a certain type of phrase, can we write it as the indicated product of simpler phrases?". It was later pointed out that we will want the simpler phrases, or factors, to be of the same type as the expression we start with. For example, the factors of a given positive integer should also be positive integers.

We are now ready to take up the problem of factoring expressions of a certain type, namely polynomials. When we write polynomials in factored form, we will want the factors themselves to be phrases of the same type, that is, we will want the individual factors to be polynomials also. For this reason it has been necessary to make sure that we can recognize a polynomial when we see one.

In one of our earlier examples we said that the factored form of

$$x^2 + 3x \qquad \text{is} \qquad x(x + 3).$$

Here we see that $x^2 + 3x$ is a polynomial. Do you see that the factors x and $x + 3$ are also polynomials?

It would have been possible to write

$$x^2 + 3x \quad \text{as} \quad \frac{1}{2}(2x^2 + 6x) \quad \text{or} \quad \frac{1}{x}(x^3 + 3x^2),$$

but in these two cases we see that the factors are not all polynomials. According to our understanding can you see that $\frac{1}{2}$ is not a polynomial? (Remember that we are still talking about polynomials over the integers.) Do you also see that $\frac{1}{x}$ is not a polynomial? Why?

In working with positive integers, we used the term prime number. This referred to an integer greater than 1 which had no proper factors. It is convenient to use the same idea in connection with the factoring of polynomials. We shall want to call a certain type of polynomial a "prime" polynomial if it satisfies certain conditions.

In line with our discussion about integers, we shall want the word "prime" to apply if the polynomial in question cannot be factored further, that is, if it cannot be written as the product of two other polynomials. Consider, then, the polynomial

$$x - 5$$

It is possible to write this as

$$(1)(x - 5) \qquad \text{or} \qquad (-1)(5 - x),$$

where in both cases we have two "factors" which are polynomials. However, since both 1 and (-1) are factors of every polynomial, we shall not consider either $(1)(x - 5)$ or $(-1)(5 - x)$ a "proper" factorization. Thus we may call $x - 5$ a prime polynomial.

$3y^2 + 6y$ is not a prime polynomial, since it can be written as $3y(y + 2)$.

$6x + 30$ is not a prime polynomial, since it can be written as $6(x + 5)$.

Is it possible to factor the expression $3y(y + 2)$ any further? Since the answer is no, we have factored $3y^2 + 6y$ completely when we write $3y(y + 2)$. In this sense we shall call $3y(y + 2)$ a prime factorization of $3y^2 + 6y$.

We noted above that $6(x + 5)$ is a factored form of $6x + 30$. We also observe that $6(x + 5)$ is not "completely" factored since it can be written as $3 \cdot 2(x + 5)$. However, we shall make an exception in the case of a factor which is itself an integer, such as 6, and agree to call an expression of the form $6(x + 5)$ a prime factorization. Similarly, we shall call

$$10(x + 4)$$

a prime factorization of $10x + 40$ even though 10 can be factored as $2 \cdot 5$.

It should be clear, however, that $6(x^2 + 5x)$ is not a prime factorization. Neither is $6x + 12$. Can you see why? What is the prime factorization of $6(x^2 + 5x)$? What is the prime factorization of $6x + 12$?

Some other examples are

Polynomial	Prime Factorization
$3a + 3b$	$3(a + b)$
$x^2 + 2x + 1$	$(x + 1)(x + 1)$
$3x^3 + 6x^2 + (-9x)$	$3x(x + 3)(x - 1)$

By multiplication and application of the distributive law, can you show that each expression on the right names the same number as the corresponding expression on the left?

Check Your Reading

1. What are the factors of $x^2 + 3x$?
2. What is a prime polynomial?
3. What does it mean to factor a polynomial completely?
4. What is meant by the term "prime factorization" of a polynomial?
5. Is the polynomial $6x + 12$ written in factored form?

Oral Exercises 13-2

1. It is possible to factor $x + 3x^2$ as $\frac{1}{3}x(3 + 9x)$ or $x(1 + 3x)$. Why do we prefer $x(1 + 3x)$?
2. Is $3x(3x + 6)$ the prime factored form of $9x^2 + 18x$? Why?
3. What is a prime factorization of $6x^2 + 30x$?
4. What is a prime factorization of $4ar^2 + 18ar^3$?

Problem Set 13-2

1. Which of the following sentences are true for all values of the variables?

   (a) $2r + 6s = 2(r + 3s)$

   (b) $81 + 162m = 81(1 + 2m)$

Problem Set 13-2
(continued)

(c) $7a + a^2 = a(7 + a)$

(d) $r^3 + 3r^2 - 5r = r(r^2 + 3r - 5)$

(e) $2q^2 + 2q = 2q(q)$

(f) $4q^2 + 8qr = 4(q + 2r)$

(g) $4a^3 - 12a^2 + 32a = 4a(a^2 - 3a + 8)$

(h) $27s + 18 = 9s(3s + 2)$

(i) $x^2 - 5x + 6 = (x - 2)(x - 3)$

(j) $x^2 + 7x + 12 = (x + 3)(x + 4)$

2. Which of the following are polynomials?

(a) $x^2 + 7x + 12$

(b) $\frac{1}{a} + 4a$

(c) $7s + 4r$

(d) $7a + 5 - 2q$

(e) $\frac{x + 2}{5}$

3. Which of the following are prime polynomials?

(a) $(a + b)$

(b) $a^2 + 3a$

(c) $x + 5$

(d) $r^2 + 5r$

(e) $r^2 + 2$

(f) $s^2 + s$

(g) $2a + 6$

(h) $7ar^2 + 8q^2$

4. Write the prime factorizations of the following expressions.

Example: $8c + 56 = 8 \cdot c + 8 \cdot 7$

$= 8(c + 7)$

(a) $16r + 24$

(b) $6a^3 + 12a^2 + 18a$

(c) $5b - 35c$

(d) $9q^2 - 27q$

(e) $4ar + 24ax + 32az$

(f) $22a^2x^2 - 33a^2x^3$

13-3. Common Monomial Factoring.

The expression

$$4st + 8st^2 + 28s^2t^3$$

is a polynomial in two variables. It can be written in various factored forms

(a) $2(2st + 4st^2 + 14s^2t^3)$

(b) $4(st + 2st^2 + 7s^2t^3)$

(c) $4s(t + 2t^2 + 7st^3)$

(d) $4st(1 + 2t + 7st^2)$

Which of these is a prime factorization? In all of these forms we have used the distributive property to "factor out" something. In (a), (b), and (c) there is still something left inside the parentheses which can be "factored out". Can you see in each case what this is? In (d), on the other hand, we have "factored out" everything we can. Form (d) is a prime factorization.

Expressions like

$$2, \qquad 4, \qquad 4s, \qquad 4st$$

are examples of what we call monomials. A monomial is a special kind of polynomial. A monomial is a polynomial in which the only indicated operation is either multiplication or taking opposites, or in which there is no indicated operation at all. For instance

$$25xyz, \qquad 7, \qquad -x, \qquad 12s^2t^3, \qquad y^2$$

are all monomials.

$$x + 5, \qquad 3y - 2, \qquad s + t - r$$

are not monomials. Why?

Thus, each of the factorizations in (a)--(d) are examples of common monomial factoring, because one of the factors is a monomial. But only one of these, (d), is a prime factorization.

As a further example of common monomial factoring, suppose we were asked to factor the polynomial

$$5r^2q - 10r^3q^2 + 15r^4q^3$$

into prime factors. This can be written as

$$5r^2q(1) + 5r^2q(-2rq) + 5r^2q(3r^2q^2) = 5r^2q(1 - 2rq + 3r^2q^2).$$

The form on the right is a prime factorization.

Check Your Reading

1. Is the expression $4(st + 2st^2 + 7s^2t^3)$ an example of prime factorization? Why?
2. What is a monomial?
3. Is a monomial also a polynomial? Explain.
4. Is $12s^2t^3$ a monomial?
5. Is $3y - 2$ a monomial?

Oral Exercises 13-3a

1. What is a prime factorization of each of these expressions?
   - (a) $3a + 3b$
   - (b) $2x^3 + 4x^2 + 8x$
   - (c) $3x^3 + 6x^2 + 9x$
   - (d) $3y^2 - 6y$
2. Use common monomial factoring to find the prime factors of the following. In each case state the monomial factor.
   - (a) $7 + 28a$
   - (b) $4x^2 - 8x$
   - (c) $5mn + 15m^2n^2$
   - (d) $7arx - 14ax^2$
3. Which of the following are monomials?
   - (a) $7x$
   - (b) $7 + x$
   - (c) $7arx$
   - (d) $7a - rx$
   - (e) $7ar^2$

Problem Set 13-3b

1. In which of the following is the right side a prime factorization of the left side?

   (a) $2x^2 - 4x = 2x(x - 2)$

   (b) $2a - 2b - 2c = 2(a - b - c)$

   (c) $4xy - 4y^2 - 8y = 2y(2x - 2y - 4)$

   (d) $7a^2x + 14ax - 21ax^2 = 7ax(a + 2 - 3x)$

   (e) $3b^3 + 6b^2 - 12b = 3b(b^2 + 2b - 4)$

   (f) $4x^4 - 10x^3 + 2x^2 = 2x(2x^3 - 5x^2 + x)$

   (g) $3x^2y - 15xy + 18y^3 = 3y(x^2 - 5x + 6y^2)$

2. Give a prime factorization of each of the following, if possible.

   (a) $3x + 6$

   (b) $8c + 24$

   (c) $2a - 18$

   (d) $4y - 35$

   (e) $6x + 9$

   (f) $8a + 16b + 23c$

   (g) $a^2 + ab$

   (h) $c^3 + c^2d$

   (i) $a^3 + a^2 + 2a$

   (j) $c^2n^4 - cn^2$

   (k) $5x^2 + 10x + 20$

   (l) $2a + 18a^2$

   (m) $6m^2 - 9m$

   (n) $2ac + 6ad + 10abd$

   (o) $x^3 - x^2 + q$

   (p) $3x^2 - 12x^3 + 6x$

   (q) $bx^2 + by^2$

   (r) $2b^3 - 6b^2 - 4b$

   (s) $3ax + 15ay - 9a^2$

   (t) $7ab^2x - 63bc^2x$

13-4. Quadratic Polynomials.

In previous chapters we learned to find products such as

$$(x + 5)(x + 3).$$

Using the properties of the operations we find that

$$\begin{aligned}(x + 5)(x + 3) &= (x + 5)x + (x + 5)(3) \\ &= (x^2 + 5x) + (3x + 15) \\ &= x^2 + (5x + 3x) + 15 \\ &= x^2 + 8x + 15.\end{aligned}$$

Check to see if you can follow each of the steps in the above process.

The above illustration makes it clear that

$(x + 5)(x + 3)$ is the factored form of $x^2 + 8x + 15$.

If we were given the polynomial $x^2 + 8x + 15$ and were told to factor this polynomial, our answer would have to be

$$(x + 5)(x + 3).$$

The question is this: How would we have been able to write the factored form

$$(x + 5)(x + 3)$$

if we had not known the factors ahead of time? We see that the process of factoring is definitely connected with that of finding products. In fact, it is really a matter of going through the steps "backwards". But this is not so easy. Have you ever been asked to say the alphabet backwards? You can do it, but it usually takes longer and requires some thinking.

In the above example the process of "going backwards" might have been easier if we could have started with the next to last step. Instead of the problem of factoring "$x^2 + 8x + 15$" suppose we had been asked to factor the polynomial

$$x^2 + 5x + 3x + 15.$$

If we look at the first two terms by themselves, we see that

$x^2 + 5x$ could be written as $(x + 5)x$

by the distributive property. Likewise,

$3x + 15$ could be written as $(x + 5)3$.

Thus, we could say that

$$x^2 + 5x + 3x + 15 = (x + 5)x + (x + 5)3.$$

If we look at the expression

$$(x + 5)x + (x + 5)3,$$

we can think of it as consisting of two parts "$(x + 5)x$" and "$(x + 5)3$". It should be clear that the expression "$(x + 5)$" in parentheses is a factor which is "common" to both parts. Suppose now that we apply the distributive property in which we think of $(x + 5)$ as a single numeral. In other words we "factor out" $(x + 5)$ as though it were a monomial factor. The result is as follows:

$$(x + 5)x + (x + 5)3 = (x + 5)(x + 3).$$

Thus we have

$$x^2 + 5x + 3x + 15 = (x + 5)(x + 3).$$

In other words, we now see how the factoring operation can be carried out once we have an expression of the form

$x^2 + ax + bx + ab$ where a and b are integers.

A few examples and exercises of this type will help strengthen the idea. Consider the following. See if you can follow the steps in each case.

$$\begin{aligned} x^2 + 3x + 2x + 6 &= (x^2 + 3x) + (2x + 6) \\ &= (x + 3)x + (x + 3)2 \\ &= (x + 3)(x + 2) \end{aligned}$$

$$\begin{aligned} 2x^2 - 10x + x - 5 &= (2x^2 - 10x) + (x - 5) \\ &= (x - 5)2x + (x - 5)1 \\ &= (x - 5)(2x + 1) \end{aligned}$$

Check Your Reading

1. How many times was the distributive property used in obtaining the result

$$(x + 5)(x + 3) = x^2 + 8x + 15?$$

Check Your Reading
(continued)

2. In the expression $(x + 5)x + (x + 5)3$ what factor is common to both terms?

3. How many terms are there in the expression $(x + 3)x + (x + 3)2$? What factor do these terms have in common?

4. Factoring is the reverse of what operation?

Problem Set 13-4a

1. Group the following polynomials into two terms.

   (a) $b^2 + b + b + 1$ (c) $a^2 - 12a + 2a - 24$

   (b) $m^2 + 2m - 3m - 6$ (d) $b^2 - 6b - 4b + 24$

2. Group the following polynomials into two terms and factor each term using common monomial factoring.

   Example: $x^2 - 3x + 5x - 15 = (x^2 - 3x) + (5x - 15)$
   $= (x - 3)x + (x - 3)5$

   (a) $c^2 + 3c + 5c + 15$

   (b) $9c^2 + 12c - 12c - 16$

   (c) $35m^2 - 28m + 15m - 12$

   (d) $2x^2 + x + 14x + 7$

3. Group the following polynomials into two terms, factor each term using common monomial factoring, and complete the factoring as in the example.

   Example: $x^2 - 3x + 5x - 15 = (x^2 - 3x) + (5x - 15)$
   $= (x - 3)x + (x - 3)5$
   $= (x - 3)(x + 5)$

   (a) $m^2 + 2m + 3m + 6$

   (b) $y^2 + 3y + 4y + 12$

   (c) $z^2 - 2z + 3z - 6$

   (d) $x^2 + 5x - 4x - 20$

Problem Set 13-4a
(continued)

4. Factor the following polynomials (as in Exercise 3).

   (a) $3x^2 + 2x + 9x + 6$

   (b) $10r^2 - 15r + 4r - 6$

   (c) $6q^2 - 10q + 9q - 15$

   (d) $8y^2 - 20y - 6y + 15$

   (e) $4m^2 + 2m + 2m + 1$

   (f) $12a^2 + 16a + 3a + 4$

5. Factor the following polynomials.

   (a) $x^2 - 4x + bx - 4b$

   (b) $5c + 5d + ac + ad$

   (c) $9ma + 6ab + 12m + 8b$

   (d) $m^2a - 2bm^2 + na - 2bn$

   (e) $5x - 5y + ax - ay$

   (f) $6s^2 + 9s + 8s + 12$

We return again to the job of factoring a polynomial such as

$$x^2 + 7x + 12.$$

Thus far we have learned how to factor this type of expression after it is written in the form

$$x^2 + ax + bx + ab.$$

We must now find a way to change the first form to the second. To accomplish this let us see how the integers a and b in the second form are related to the 7 and the 12 in the first. A study of this particular case will give us some general ideas, which we can apply to other examples.

Using the distributive property we can write

$$x^2 + ax + bx + ab$$

as

$$x^2 + (a + b)x + ab.$$

If we now examine the two expressions:

$$x^2 + (a + b)x + ab \qquad \text{and} \qquad x^2 + 7x + 12,$$

we see that the product ab represents 12 and the sum $(a + b)$ represents 7. Thus our problem is to find two integers whose sum is 7 and whose product is 12. What are these two integers? If we consider all pairs of integers whose product is 12, we come up with these choices:

$$1 \cdot 12 \qquad 2 \cdot 6 \qquad 3 \cdot 4$$

Do you see which pair has a sum of 7? The correct choice will show us that the form

$$x^2 + ax + bx + ab$$

for this example is

$$x^2 + 3x + 4x + 12.$$

By previous methods we can now show that

$$x^2 + 3x + 4x + 12 = (x + 3)(x + 4)$$

which means that

$$x^2 + 7x + 12 = (x + 3)(x + 4).$$

Is this a prime factorization?

The problem of finding two integers with a specified product and a specified sum is a familiar one, which we studied in Chapter 11. The method involved prime factorization. When the sums and products are fairly small, the answers can often be obtained by inspection or some form of systematic guessing. On the other hand the use of prime factors can be of help in dealing with larger numbers. The following examples will lead to greater understanding. See if you can follow the reasoning in each case.

Example 1. Factor $x^2 + 10x + 24$.

For the form $x^2 + ax + bx + ab$ we want integers a and b whose product is 24 and whose sum is 10. Possible choices:

$1\cdot24 \qquad 2\cdot12 \qquad 3\cdot8 \qquad 4\cdot6$

Which do you take? The correct choice should give you the following result.

$$\begin{aligned} x^2 + 10x + 24 &= x^2 + 4x + 6x + 24 \\ &= (x + 4)x + (x + 4)6 \\ &= (x + 4)(x + 6). \end{aligned}$$

Example 2. Factor $x^2 - 17x + 16$

Here we might find it helpful to write the polynomial

$$x^2 + (-17)x + 16.$$

This tells us that we are looking for two integers with product 16 and sum (-17). For the product 16 we recall that the factors must be either both positive or both negative. Since the sum must be negative, that is, (-17), what do we conclude? Possible choices of two negative factors are

$(-1)(-16) \qquad (-4)(-4) \qquad (-2)(-8).$

What is your choice? Do you see that

$$x^2 - 17x + 16 = (x - 1)(x - 16).$$

The intermediate steps have been omitted. You should, however, write them down to be sure that you understand them.

Example 3. Find the prime factorization of $w^2 - w - 20$.

This we can write as

$$w^2 + (-1)w + (-20).$$

Our integers must have a product of (-20) and a sum of (-1). The product is negative. What does this tell us about the factors? Do you see that one must be positive and the other negative? The sum (-1)

is also negative. This tells us that the negative factor has the greater absolute value. With this information in mind we can limit our choices to the following pairs:

$$(1)(-20) \qquad (2)(-10) \qquad (4)(-5)$$

Only one of these pairs has a sum equal to (-1). Do you see which one it is? The proper choice gives us

$$w^2 - w - 20 = (w + 4)(w - 5).$$

Again you should complete the intermediate steps.

Example 4. Factor $y^2 - 27y - 90$.

In this example we shall use prime factors to illustrate the method. We wish integers with product (-90) and sum (-27). The prime factorization of (-90) with an additional factor 1 gives us

$$90 = 1 \cdot 2 \cdot 3^2 \cdot 5.$$

Before thinking of possible combinations we should note two things. The negative product (-90) requires one positive and one negative factor. The sum (-27) tells us that the negative factor has the greater absolute value. To select our choice we group factors in the following way:

$\underline{1}$	$\underline{-(2 \cdot 3^2 \cdot 5)}$	$\underline{2}$	$\underline{-(3^2 \cdot 5)}$
$\underline{3}$	$\underline{-(2 \cdot 3 \cdot 5)}$	$\underline{5}$	$\underline{-(3^2 \cdot 2)}$
$\underline{2 \cdot 3}$	$\underline{-(3 \cdot 5)}$	$\underline{3^2}$	$\underline{-(2 \cdot 5)}$

The correct combination gives us

$$y^2 - 27y - 90 = (y + 3)(y - 30).$$

Check Your Reading

1. If we think of the expression $x^2 + 7x + 12$ as having the form $x^2 + (a + b)x + ab$, what numbers do $a + b$ and ab represent?

2. Give all pairs of integers whose product is 12. What are all possible sums of these factors?

3. Give the pair of integers whose product is 12 and whose sum is 7.

Oral Exercises 13-4b

1. For each of the following pairs of numbers find two integers such that their product is the first number of the pair and their sum is the second number of the pair.

   (a) 10, 7
   (b) 10, -7
   (c) 8, 6
   (d) 8, -6
   (e) -8, -2
   (f) -8, 2
   (g) 8, 9
   (h) 8, -9
   (i) 1, 2
   (j) 1, -2
   (k) -2, -1
   (l) -2, 1

2. Express each of the following in the form $x^2 + ax + bx + ab$.
   Example: $x^2 - x - 20$. We want integers a and b such that their product is -20 and their sum is -1. Thus

$$x^2 - x - 20 = x^2 - 5x + 4x - 20.$$

   (a) $x^2 + 6x + 8$
   (b) $x^2 + 9x + 8$
   (c) $x^2 - 6x + 8$
   (d) $x^2 - 7x - 8$
   (e) $x^2 + 2x - 8$
   (f) $x^2 - 2x - 8$
   (g) $x^2 + 7x + 12$
   (h) $x^2 - 8x + 12$
   (i) $x^2 + 13x + 12$
   (j) $x^2 - x - 12$
   (k) $x^2 + 4x - 12$
   (l) $x^2 - 11x - 12$

Problem Set 13-4b

1. Express each of the following in the form $x^2 + ax + bx + ab$:

   (a) $x^2 + 8x + 15$
   (b) $r^2 - 2r - 15$
   (c) $s^2 + 8s - 9$
   (d) $t^2 + 5t + 6$
   (e) $w^2 - 10w - 11$
   (f) $m^2 - 4m - 12$
   (g) $a^2 - 7a + 12$
   (h) $y^2 + 10y + 24$
   (i) $z^2 - 11z + 30$
   (j) $c^2 - 2c - 24$

2. Factor each of the following polynomials:

   (a) $t^2 + 12t + 35$
   (b) $w^2 - 7w + 10$
   (c) $r^2 - 13r + 22$
   (d) $a^2 - 18a + 77$
   (e) $m^2 + 3m - 18$
   (f) $z^2 + 4z - 21$
   (g) $a^2 + 6a - 55$
   (h) $y^2 - 7y + 12$
   (i) $x^2 - 3x - 4$
   (j) $b^2 - 9b + 8$

3. Factor each of the following polynomials:

   (a) $y^2 - 7y - 18$
   (b) $a^2 + 4a - 5$
   (c) $x^2 - 7x - 30$
   (d) $w^2 - 8w + 16$
   (e) $w^2 - 6w - 7$
   (f) $k^2 + 6k - 7$
   (g) $x^2 - 15x - 16$
   (h) $x^2 - 11x + 18$
   (i) $m^2 + 0 \cdot m - 16$
   (j) $t^2 + 2t + 1$

A polynomial such as

$$4x^2 - 3x + 2$$

is called a polynomial of the *second degree*, or *quadratic polynomial*. In this polynomial

$4x^2$ is called the *second degree term*,
$-3x$ is called the *first degree term*,
and 2 is called the *constant term*.

We also say that

4 is the coefficient of x^2 in the second degree term,

and -3 is the coefficient of x in the first degree term.

The constant term 2 can also be called a coefficient.

The degree of a polynomial in one variable is given by its term of highest degree. Thus,

$5x + 7$ is a first degree polynomial, and

$2x^3 + 6x^2 - 1$ is a third degree polynomial.

In this same sense we can call an expression in the form of a non-zero integer such as 7 or -10, a polynomial of degree zero.

The factoring problems which we have just been studying involved quadratic polynomials of the form

$$x^2 + 7x + 10,$$

in which the coefficient of the second degree term is 1. Suppose, now we were asked to factor a quadratic polynomial such as

$$6x^2 + 7x + 2,$$

in which the coefficient of the second degree term(sometimes called the leading coefficient) is greater than 1. How would we go about this?

Up to now we have been able to write our answers in the form

$$(x + m)(x + n),$$

where m and n are integers and the first term in each factor is the variable itself. However, we see that this will not be possible in our present problem, since multiplication of the above factors will give us a second degree term of x^2. Because our second degree term is $6x^2$, we shall need factors whose first terms are $6x$ and x, or $3x$ and $2x$. Are there any other possibilities?

The constant term of $6x^2 + 7x + 2$ is 2. Thus our second terms in each factor must have a product of 2. Do you see why? With these facts in mind, we can list the following choices:

$$(6x + 1)(x + 2)$$

$$(6x + 2)(x + 1)$$

$$(3x + 1)(2x + 2)$$

$$(2x + 1)(3x + 2)$$

Multiplication of the first two factors gives us

$$\begin{aligned}(6x + 1)(x + 2) &= (6x + 1)x + (6x + 1)2\\ &= 6x^2 + x + 12x + 2\\ &= 6x^2 + 13x + 2\end{aligned}$$

Everything is all right except the first degree term. We have $13x$. We want $7x$. Find the other products! Do you see that the last one gives us

$$\begin{aligned}(2x + 1)3x + (2x + 1)2 &= 6x^2 + 3x + 4x + 2\\ &= 6x^2 + 7x + 2?\end{aligned}$$

We have found the correct factors by a sort of "trial and error" process. You might even have guessed the right combination the first time. However, there are disadvantages to a trial and error method, especially if the number of choices is fairly large.

Let us look instead at the ideas we used for the previous type (with a leading coefficient of 1) and see if we can "extend" these ideas to include the new type.

In factoring polynomials of the previous type we first wrote a second form consisting of four terms with ax and bx as the second and third terms. Let us now write a second form for the polynomial

$$6x^2 + 7x + 2$$

as follows: $6x^2 + ax + bx + 2$

Our task is to find the correct integers a and b. We can then factor this form in much the same way as before.

In the first place it should be clear that we want the sum of a and b to be 7. So far the process is the same as before. But what about the product $(a)(b)$? To answer this it will help to examine once again the multiplication operation, which is the reverse of factoring. Let's use the answer we got by trial and error. We recall that this was $(2x + 1)(3x + 2)$. Omitting the first step we see that

$$(2x + 1)(3x + 2) = 6x^2 + 3x + 4x + 2$$

This tells us that a should be 3 and b should be 4. Thus we see that in the correct form $(a)(b) = (3)(4) = 12$. In other words, the product ab is not equal to the constant term 2 as in the previous type. However, note that if we multiply the constant term 2 by the leading coefficient 6, we also get 12. This suggests an important question. Is it true that the product of a and b should be equal to the constant term multiplied by the leading coefficient in every case? If so, then we can "extend" our previous method of factoring to include the new type as follows: To factor

$$6x^2 + 7x + 2$$

write the form $\quad 6x^2 + ax + bx + 2.$

Determine integers a and b such that

$$a + b = 7$$

and $\quad (a)(b) = (6)(2) = 12.$

In this case, we have found that $a = 3$ and $b = 4$. To factor, we note that

$$\begin{aligned} 6x^2 + 3x + 4x + 2 &= (2x + 1)3x + (2x + 1)2 \\ &= (2x + 1)(3x + 2). \end{aligned}$$

We have not answered the question as to whether or not this will "work" every time. Let's look at one more example, then see if we can arrive at a satisfactory proof for the general case.

First, see if we can factor

$$8x^2 + 10x + 3.$$

Second form: $\quad 8x^2 + ax + bx + 3.$

We must now find integers a and b whose sum is 10 and whose product is 24. There are several possibilities, but a little thought will suggest the integers 4 and 6. Thus we have

$$\begin{aligned} 8x^2 + 4x + 6x + 3 &= (2x + 1)4x + (2x + 1)3 \\ &= (2x + 1)(4x + 3). \end{aligned}$$

If we multiply the last two factors, we obtain the original polynomial

$$8x^2 + 10x + 3.$$

We shall now examine the method to find out if it works for all cases. Before doing this, however, we must realize that when we speak of a method "working" we are assuming that the quadratic polynomial in question can be factored to begin with. There are, of course, many quadratic polynomials which cannot be factored over the integers. For example the polynomial

$$2x^2 + 3x + 4$$

cannot be expressed as the product of two polynomials over the integers. (You may try to find them, but don't work at it too long!) In other words, this is an example of a prime polynomial.

But now let us assume we have a polynomial which can be factored, and that the factored form is

$$(cx + d)(ex + f)$$

where the letters c, d, e, and f represent integers. To form the product we see that

$$\begin{aligned}(cx + d)(ex + f) &= (cx + d)ex + (cx + d)f \\ &= cex^2 + dex + cfx + df.\end{aligned}$$

Here the steps have been carried out exactly as in the previous illustrations.

We now wish to find out whether or not our so-called method works. In the above examples we suggested that the product of a and b in the second form of four terms should be equal to the constant term multiplied by the leading coefficient. Now examine the polynomial

$$cex^2 + dex + cfx + df.$$

Do you see that de and cf take the place of a and b respectively, and the constant term is df and the leading coefficient is ce? It now remains to be seen whether or not the two products are equal. We ask the question, is the sentence

$$(de)(cf) = (ce)(df)$$

true for all integers c, d, e, f? Because of the associative and commutative properties of multiplication, we see that the answer is yes.

A few more examples should serve to fix the ideas firmly in your mind.

Example 1. Factor $6x^2 - 11x + 4$.

Second form: $6x^2 + ax + bx + 4$

Requirements for a and b: $a + b = (-11)$

$(a)(b) = 24$

Since the sum is negative and the product is positive, we know that both a and b are negative. Will (-3) and (-8) give us the correct results?

$$\begin{aligned}6x^2 - 3x - 8x + 4 &= (2x - 1)3x + (2x - 1)(-4)\\ &= (2x - 1)(3x - 4).\end{aligned}$$

Example 2. Factor $3x^2 - 5x - 2$.

Second form: $3x^2 + ax + bx - 2$

Requirements for a and b: $a + b = (-5)$

$(a)(b) = (-6)$

We see that (-6) and 1 will satisfy these requirements. Thus

$$\begin{aligned}3x^2 - 6x + x - 2 &= (x - 2)3x + (x - 2)1\\ &= (x - 2)(3x + 1).\end{aligned}$$

Check Your Reading

1. What is the degree of the polynomial $4x^2 + 3x + 2$? What is the coefficient of x in the first degree term?
2. What is the leading coefficient and the constant term of the polynomial $6x^2 + 7x + 2$?

Check Your Reading
(continued)

3. What is the sum and the product of a and b when $6x^2 + 7x + 2$ is considered in the form $6x^2 + ax + bx + 2$? How do you determine what the sum is? How do you determine what the product is?

4. If the polynomial $8x^2 + 10x + 3$ is considered in the form $8x^2 + ax + bx + 3$, what is the product of the leading coefficient and the constant term? What is the product of a and b?

Oral Exercises 13-4c

1. Given the following polynomials:

   $2x^2 - 3x$, $x^2 + 5$, $x^2 - 2x + 5$, $3x - 7$, $2x^2$,
   $5x$, $3 + 2x + 5x^2$, $5 - 2x$, $9 - x^2$, $x^3 + 3x^2$

   (a) Which are polynomials of the first degree?

   (b) Which are polynomials of the second degree?

   (c) Which polynomials have a second degree term?

   (d) Which polynomials are not quadratic polynomials?

   (e) Which polynomials have a second degree term and a first degree term?

   (f) Which polynomials have a constant term?

   (g) For each of the polynomials which has a first degree term, name its coefficient.

   (h) For each of the quadratic polynomials, name the leading coefficient.

2. Consider each of the following polynomials in the form $3x^2 + ax + bx + 8$. Find a and b for each.

   (a) $3x^2 - 11x + 8$
   (b) $3x^2 + 10x + 8$
   (c) $3x^2 - 23x - 8$
   (d) $3x^2 + 10x - 8$
   (e) $3x^2 + 5x - 8$
   (f) $3x^2 - 2x - 8$

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Oral Exercises 13-4c
(continued)

(g) $3x^2 + 25x + 8$

(h) $3x^2 - 14x + 3$

Problem Set 13-4c

1. Factor the following polynomials:

(a) $2y^2 + 7y + 3$

(b) $3y^2 + 7y + 2$

(c) $4a^2 + 4a + 1$

(d) $3n^2 - 4n + 1$

(e) $3x^2 - 5x + 2$

(f) $8x^2 - 14x + 3$

(g) $h^2 - 10h - 24$

(h) $7x^2 + 6x - 13$

(i) $3x^2 + 20x - 7$

(j) $5x^2 - 2x - 7$

(k) $2h^2 - 9h - 5$

(l) $4u^2 - 12u + 9$

(m) $v^2 + 13v + 36$

(n) $6x^2 + 23x + 21$

(o) $14t^2 + 39t + 10$

(p) $35y^2 - 22y + 3$

2. Factor.

(a) $8x^2 + 121x + 15$

(b) $8x^2 - 62x + 15$

(c) $8x^2 + 26x - 15$

(d) $8x^2 + 37x - 15$

(e) $8x^2 - 43x + 15$

(f) $8x^2 + 22x + 15$

(g) $8x^2 - 14x - 15$

(h) $8x^2 - 19x - 15$

(i) $8x^2 - 58x - 15$

(j) $8x^2 + 7x - 15$

(k) $8x^2 + 2x - 15$

(l) $8x^2 + 29x + 15$

(m) $8x^2 - 119x - 15$

(n) $8x^2 + 34x + 15$

(o) $8x^2 - 23x + 15$

(p) $8x^2 + 26x + 15$

Problem Set 13-4c
(continued)

3. Factor.

(a) $4x^2 + 0 \cdot x - 9$

(b) $4h^2 + 12h + 9$

(c) $z^2 + 0 \cdot z - 25$

(d) $v^2 + 2v + 1$

(e) $2y^2 - y - 36$

(f) $4y^2 + 0 \cdot y - 36$

(g) $4y^2 + 24y + 36$

(h) $x^2 + 11x + 30$

(i) $5x^2 + 2x - 72$

(j) $5x^2 - 66x + 72$

13-5. Difference of Squares.

In the previous section we have learned a general method for factoring quadratic polynomials. Thus we have, in a sense, completed the job with respect to this type of expression.

There are, however, special cases for which the factoring process may be greatly simplified. We shall study two such types. Consider the following quadratic polynomials:

$$x^2 - 9, \qquad N^2 - 16, \qquad 4y^2 - 25$$

What are the special features which all of these have in common? First we note that the first degree term is missing in each case; that is, the coefficient of this term is zero. We also see that each of the polynomials shows an indicated operation of subtraction. There is a simple and direct way of factoring this type. To discover this let us first factor using the method of the previous section.

For

$$x^2 - 9$$

the second form is $x^2 + ax + bx - 9$.

We seek integers a and b whose product is (-9). What about the sum? Do you see that this is zero? That is,

$$(a)(b) = (-9)$$

and $\quad a + b = 0.$

From properties we have studied in Chapter 7, we know that this last statement is true if and only if b is the opposite of a. In other words, our two requirements on a and b for this special type are

$$\begin{aligned} (a)(b) &= (-9) \\ a &= (-b). \end{aligned}$$

Do you see that these two requirements will be met if $a = 3$ and $b = (-3)$? Therefore, we now have

$$\begin{aligned} x^2 + 3x - 3x - 9 &= (x + 3)x + (x + 3)(-3) \\ &= (x + 3)(x - 3). \end{aligned}$$

Likewise for

$$N^2 - 16$$

we see that in the form

$$N^2 + aN + bN - 16, \quad a = 4 \text{ and } b = -4$$

will meet the requirements. Hence

$$\begin{aligned} N^2 + 4N - 4N - 16 &= (N + 4)(N) + (N + 4)(-4) \\ &= (N + 4)(N - 4). \end{aligned}$$

For the polynomial

$$4y^2 - 25$$

we write

$$4y^2 + ay + by - 25.$$

In this case we note that $a + b = 0$ and that $(a)(b) = (4)(-25) = (-100)$. The requirements will be satisfied if $a = 10$ and $b = -10$. Thus we complete the factoring as follows:

$$\begin{aligned} 4y^2 + 10y - 10y - 25 &= (2y + 5)2y + (2y + 5)(-5) \\ &= (2y + 5)(2y - 5). \end{aligned}$$

Let us now put these three results together to see if we can figure out a short cut for the factoring process.

$$\begin{aligned} x^2 - 9 &= (x + 3)(x - 3) \\ N^2 - 16 &= (N + 4)(N - 4) \\ 4y^2 - 25 &= (2y + 5)(2y - 5) \end{aligned}$$

There are some things which all of these results have in common. The first factor in each case represents a sum and the second factor represents a difference. In other respects the factors are the same. Do you see how the terms are formed in each factor? In the last example the polynomial $4y^2 - 25$ could be written as $(2y)^2 - (5)^2$. This may help us to understand how the factors $(2y + 5)(2y - 5)$ are formed. A few additional multiplications should clear up any doubts. These examples will indicate the factoring process in reverse.

$$\begin{aligned}(y + 6)(y - 6) &= (y + 6)y + (y + 6)(-6)\\ &= y^2 + 6y - 6y - 36\\ &= y^2 - 36\end{aligned}$$

$$\begin{aligned}(3x + 7)(3x - 7) &= (3x + 7)3x + (3x + 7)(-7)\\ &= 9x^2 + 21x - 21x - 49\\ &= 9x^2 - 49.\end{aligned}$$

We should now be in a position to factor polynomials of this type without having to write the intermediate steps. Do you see, for example, that

$$25w^2 - 9 = (5w + 3)(5w - 3) ?$$

It should be clear that this type of factoring is possible only when the leading coefficient and the absolute value of the constant term are squares of integers. Thus the quadratic polynomials

$$x^2 - 8 \qquad \text{and} \qquad 3y^2 - 16$$

cannot be factored over the integers. Do you also see that

$$x^2 + 9$$

cannot be factored even though 9 is the square of an integer? Why is this?

Thus far we have been discussing quadratic polynomials in one variable. We can also show that

$$9x^2 - 4y^2 = (3x + 2y)(3x - 2y),$$

where more than one variable appears in the polynomial.

Likewise

$$25s^2t^2 - 36r^2 = (5st + 6r)(5st - 6r)$$

In this case we have a polynomial in three variables. It will help to multiply the factors in each of the above examples to show that the results are correct.

Because of the special form in which they appear it is customary to refer to polynomials of the type we have studied in this section as the

difference of two squares.

Check Your Reading

1. When $x^2 - 9$ is considered in the form $x^2 + ax + bx - 9$, what is the product of a and b? What is the sum of a and b?
2. Complete the sentence: $a + b = 0$ if and only if b is the ____________ of a.
3. Complete the sentence: $4y^2 - 25 = (\quad)^2 - (5)^2$.
4. Why is it that $x^2 - 8$ and $3y^2 - 16$ cannot be factored over the integers?
5. Does $x^2 + 9$ fall into the category referred to as the difference of two squares? Why?

Oral Exercises 13-5

1. Factor each of the following polynomials.
   - (a) $n^2 - 25$
   - (b) $t^2 - 36$
   - (c) $49 - a^2$
   - (d) $4x^2 - 25$
   - (e) $m^2 - n^2$
   - (f) $4x^2 - 1$
   - (g) $a^2b^2 - 4$
   - (h) $9z^2 - 25$
   - (i) $16a^2 - 4b^2$
   - (j) $1 - 4t^2$

Oral Exercises 13-5
(continued)

2. Express each of the following products as a difference of squares.

   (a) $(t + 1)(t - 1)$
   (b) $(x + 6)(x - 6)$
   (c) $(s + 10)(s - 10)$
   (d) $(x - 7)(x + 7)$
   (e) $(2x + 1)(2x - 1)$
   (f) $(3m - 2)(3m + 2)$
   (g) $(u + v)(u - v)$
   (h) $(st - u)(st + u)$
   (i) $(1 + 2k)(1 - 2k)$
   (j) $(w - 15)(w + 15)$

Problem Set 13-5

1. Factor each of the following polynomials.

   (a) $n^2 - 16$
   (b) $x^2 - 25$
   (c) $t^2 - 81$
   (d) $81 - t^2$
   (e) $64 - s^2$
   (f) $w^2 - 1$
   (g) $r^2s^2 - 4$
   (h) $9r^2s^2 - 4$
   (i) $25n^2 - 1$
   (j) $16 - 49k^2$

2. Write each of the following indicated products as the difference of squares.

   Example: $(x + 3)(x - 3) = (x)^2 - (3)^2$
   $= x^2 - 9.$

   (a) $(n + 4)(n - 4)$
   (b) $(t + 1)(t - 1)$
   (c) $(x + 6)(x - 6)$
   (d) $(s + 10)(s - 10)$
   (e) $(x - 7)(x + 7)$
   (f) $(w - 15)(w + 15)$
   (g) $(x - 20)(x + 20)$
   (h) $(mn + 1)(mn - 1)$
   (i) $(2mn + x)(2mn - x)$
   (j) $(2x + 3)(2x - 3)$

Problem Set 13-5
(continued)

3. Factor each of the following polynomials:

(a) $x^2 - y^2$

(b) $x^2 - 2x - 15$

(c) $5y^2 - 17y + 6$

(d) $5a^2 + 5a$

(e) $m^2 - 2m + 1$

(f) $9a^2b^2 - 25$

(g) $6z - 18z^2$

(h) $2t^2 + 3t - 5$

(i) $1 - 144t^2$

(j) $9x^2 + 81$

(k) $u^2 + 18u + 17$

(l) $a^2b + b^2a$

4. Try to write a common name for each of the following indicated products by doing the work without paper and pencil.

Example: $(39)(41)$

Think $(39)(41) = (40 - 1)(40 + 1)$
$= 40^2 - 1^2$
$= 1600 - 1$

Write $(39)(41) = 1599$.

(a) 18×22

(b) $(28)(32)$

(c) $(29)(31)$

(d) $48 \cdot 52$

(e) $(37)(43)$

(f) $(17x)(23)$

(g) $38 \cdot 42x$

(h) $(49x)(51y)$

(i) $(101)(99)$

(j) $(6)(6)(4)(11)$

*5. Factor the following polynomials over the integers if possible.

(a) $n^2 - 1$

(b) $(x + 1)^2 - 1$

(c) $(x + 1)^2 - a^2$

(d) $(x + 1)^2 - (a - 1)^2$

(e) $(m + n)^2 - (m - n)^2$

(f) $(x^2 - y^2) - (x - y)$

*6. (a) Can 899 be a prime number? $899 = 30^2 - 1^2$
$= (30 - 1)(30 + 1)$
Is it prime?

(b) Is 1591 a prime number?

13-6. Perfect Squares.

A second special type of quadratic polynomial can also be factored in a simple way. Consider the polynomials

$$x^2 + 6x + 9, \qquad y^2 - 10y + 25, \qquad 9N^2 + 24N + 16$$

The "special features" of this type are not as easy to recognize. The factoring process, however, will point these out. We shall begin with the so-called general method.

For

$$x^2 + 6x + 9$$

the form $x^2 + ax + bx + 9$ requires that

$$(a)(b) = 9$$

and $a + b = 6.$

These requirements are met if $a = 3$ and $b = 3$. Are there any other possibilities among the integers? Factoring as before, we obtain

$$\begin{aligned} x^2 + 3x + 3x + 9 &= (x + 3)x + (x + 3)3 \\ &= (x + 3)(x + 3) \end{aligned}$$

Next consider

$$y^2 - 10y + 25.$$

For the form $y^2 + ay + by + 25$ we see that $a = (-5)$ and $b = (-5)$. To continue factoring we write

$$\begin{aligned} y^2 - 5y - 5y + 25 &= (y - 5)y + (y - 5)(-5) \\ &= (y - 5)(y - 5). \end{aligned}$$

To factor

$$9N^2 + 24N + 16$$

we observe that the form $9N^2 + aN + bN + 16$ requires that

$$a + b = 24 \qquad \text{and} \qquad (a)(b) = (9)(16) = 144.$$

The requirements are fulfilled if $a = 12$ and $b = 12$. Therefore

$$\begin{aligned} 9N^2 + 12N + 12N + 16 &= (3N + 4)3N + (3N + 4)4 \\ &= (3N + 4)(3N + 4) \end{aligned}$$

Summarizing these results, we have

$$x^2 + 6x + 9 = (x + 3)(x + 3)$$
$$y^2 - 10y + 25 = (y - 5)(y - 5)$$
$$9N^2 + 24N + 16 = (3N + 4)(3N + 4)$$

In each case the two factors are the same. In fact, the results may be written $(x + 3)^2$, $(y - 5)^2$, and $(3N + 4)^2$. For this reason the general name given to this type of polynomial is perfect square.

There is little or no difficulty involved in determining the factored form, as can be seen. The problem is really one of recognition. How can we identify this type? The answer to this question can be obtained by examining the special nature of a and b in the second form. Since a and b are equal, it follows that $a + b = 2a$ and $(a)(b) = a^2$. In other words the requirements on a and b may be stated in terms of a alone. To make this clear, let's look again at the examples:

For $x^2 + 6x + 9$ we note that $2a = 6$, hence $a = 3$. If our polynomial is of the special type under consideration, then

a^2 must be equal to 9.

For $y^2 - 10y + 25$ we note that $2a = (-10)$, hence $a = (-5)$.

Does $a^2 = 25$?

Finally for $9N^2 + 24N + 16$ we see that $2a = 24$, hence $a = 12$. What must a^2 be equal to? We see that $a^2 = (9)(16) = 144$.

From these ideas we can figure out a way to determine whether or not a given quadratic polynomial is a perfect square. We first note that for perfect squares the leading coefficient and the constant term must both be squares of integers. (Don't forget that 1 is the square of 1.) If our polynomial satisfies this condition, we can then apply the following test. We divide the coefficient of the first degree term by 2. If the square of the resulting number is equal to the product of the leading coefficient and the constant term, then the given polynomial is a perfect square.

This test can be described in symbols. The polynomial

$$Ax^2 + Bx + C$$

is a perfect square if

$$(\frac{B}{2})^2 = AC.$$

Once we know that a polynomial is a perfect square, the factoring is simple. For example

$$x^2 + 12x + 36 = (x + 6)(x + 6)$$

$$N^2 - 14N + 49 = (N - 7)(N - 7)$$

$$4y^2 - 20y + 25 = (2y - 5)(2y - 5)$$

How do we know these are perfect squares? Do you see that

$$\frac{12}{2} = 6 \qquad \text{and} \qquad 6^2 = 36?$$

Also

$$\frac{(-20)}{2} = (-10) \quad \text{and} \quad (-10)^2 = (4)(25).$$

Test the second example yourself!

Suppose we were given the expression

$$x^2 - 18x.$$

What constant term should be added to make this polynomial a perfect square? We see that

$$\frac{(-18)}{2} = (-9) \qquad \text{and} \qquad (-9)^2 = 81.$$

Thus

$$x^2 - 18x + 81$$

is a perfect square. What constant should be added to make the polynomial

$$9x^2 + 24x$$

a perfect square? Here we note that

$$\frac{24}{2} = 12 \qquad \text{and} \qquad 12^2 = 144.$$

In symbols the question becomes

$$144 = (9)(?)$$

Do you see that the number in question is 16? Therefore

$$9x^2 + 24x + 16$$

is a perfect square. What is the factored form? To determine this quickly, it may help to think of the polynomial as

$$(3x)^2 + 24x + (4)^2,$$

which suggests the factors

$$(3x + 4)(3x + 4).$$

In certain situations it may be necessary to "factor out" a common monomial before applying any of the methods of the last three sections. The following examples will illustrate this.

Example 1. $5x^2 - 35x + 50 = 5(x^2 - 7x + 10)$

$$= 5(x - 2)(x - 5)$$

Example 2. $12y^2 - 75 = 3(4y^2 - 25)$

$$= 3(2y + 5)(2y - 5)$$

Example 3. $7rx^2 - 42rx + 63r = 7r(x^2 - 6x + 9)$

$$= 7r(x - 3)(x - 3)$$

As a general principle it is always wise to check for a common monomial factor before attempting to factor by other methods.

Check Your Reading

1. When $x^2 + 6x + 9$ is considered in the form $x^2 + ax + bx + 9$ what is the sum of a and b? What is the product of a and b? What are the values of a and b?
2. Complete the sentence: If a polynomial is a perfect square, the leading coefficient and the constant term are both ____________________.
3. Complete the sentence: If a polynomial of the form $Ax^2 + Bx + C$ is a perfect square, then $(\frac{B}{2})^2 =$ _____.

Check Your Reading
(continued)

4. What number should be added to $x^2 - 18x$ to form a perfect square? Explain how the number is obtained.

5. What number should be added to $9x^2 + 24x$ to form a perfect square? Explain how the number is obtained.

6. Complete the sentence: It is always wise to check for a ________________ before attempting to factor by other methods.

Oral Exercises 13-6

1. Which of the following polynomials are perfect squares?

   (a) $n^2 + 8n + 16$
   (b) $n^2 + 16$
   (c) $x^2 - 5x + 25$
   (d) $5x^2$
   (e) $t^2 - 20t + 100$
   (f) $36b^2$
   (g) $n^2 - 25$
   (h) $4x^2 + 4x + 1$

2. In each of the following incomplete expressions, what must be placed in the parentheses in order to make the resulting expression a perfect square?

   (a) $x^2 + 6x + (\ )$
   (b) $n^2 - 10n + (\ \ )$
   (c) $x^2 + 4x + (\ \ )$

3. Factor the following.

   (a) $x^2 + 8x + 16$
   (b) $n^2 - 6n + 9$
   (c) $x^2 + 2(xy) + y^2$
   (d) $4y^2 + 16y + 16$
   (e) $16m^2 + 8m + 1$
   (f) $4a^2 + 4ab + b^2$

Problem Set 13-6

1. For each of the following polynomials, answer "yes" or "no" to the question, "Is the polynomial a perfect square?".

   (a) $x^2 + 6x + 9$

   (b) $x^2 + 6x + 36$

   (c) $c^2 + 36$

   (d) $n^2 - 10n + 100$

   (e) $n^2 - 10n + 25$

   (f) $n^2 - 64$

   (g) $c^2 + 2cd + d^2$

   (h) $c^2 + cd + d^2$

   (i) $49x^2$

   (j) $4x^2 + 2x + 1$

   (k) $4x^2 + 12xy + 9y^2$

   (l) $(x - 1)^2 - 6(x - 1) + 9$

2. Complete each of the following so that the resulting polynomial is a perfect square. (This process is often called "completing the square".)

   (a) $x^2 - 8x + (\quad)$

   (b) $x^2 + 8x + (\quad)$

   (c) $n^2 + 2n + (\quad)$

   (d) $t^2 + 10t + (\quad)$

   (e) $y^2 - 16y + (\quad)$

   (f) $x^2 + (\quad) + 16$

   (g) $y^2 + (\quad) + 144$

   (h) $9a^2 + 6a + (\quad)$

   (i) $u^2 - (\quad) + 25$

   (j) $a^2 + 12a + (\quad)$

   (k) $4s^2 + 4st + (\quad)$

   (l) $(\quad) + 6xy + 9y^2$

   (m) $4s^2 + (\quad) + 9$

   (n) $(\quad) + 40v + 25$

   (o) $49x^2 - (\quad) + 16y^2$

   (p) $(v + 1)^2 + 4(v + 1) + (\quad)$

3. Factor each of the following polynomials.

   (a) $x^2 + 12x + 36$

   (b) $4s^2 - 12s + 9$

   (c) $2m^2 + 12m + 72$

   (d) $3u^2 + 6u + 3$

   (e) $v^2 - 2vt + t^2$

   (f) $36k^2 - 12k + 1$

   (g) $12t^2 + 36t + 27$

   (h) $a^3 - 2a^2b + ab^2$

   (i) $8t^2 + 8t + 2$

   (j) $x^2 - 20x + 100$

Problem Set 13-6
(continued)

4. Previous exercises have pointed out the following:

$$x^2 + 2ax + a^2 = (x + a)^2$$
$$x^2 - 2ax + a^2 = (x - a)^2.$$

This result gives us a rule for squaring expressions like $x + a$ and $x - a$ without the usual multiplication process. Perform the indicated operation for each of the following.

Example: $(x + 3)^2 = x^2 + 2(3)x + (3)^2$
$= x^2 + 6x + 9$

(a) $(x + 2)^2$	(d) $(x - 7)^2$
(b) $(a - 3)^2$	(e) $(x - y)^2$
(c) $(y + 10)^2$	(f) $(r + 6)^2$

5. Using the fact that the sentences "$(a + b)^2 = a^2 + 2ab + b^2$" and "$(a - b)^2 = a^2 - 2ab + b^2$" are true for any real numbers a and b, some indicated products of numbers can be simplified easily. Try to write a common name for each of the indicated products below.

Example: $(31)^2 = (30 + 1)^2$
$= (30)^2 + 2(30)(1) + (1)^2$
$= 900 + 60 + 1$
$= 961$

(a) $(21)^2$	(f) $(51)^2$
(b) $(41)(41)$	(g) $(49)^2$
(c) $(19)^2$	(h) $(99)^2$
(d) $(39)^2$	(i) $(101)(101)$
(e) $(29)(29)$	

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Problem Set 13-6
(continued)

6. Factor each of the following polynomials.

(a) $y^2 - 8y + 12$

(b) $x^2 + 6x + 9$

(c) $am^2 + an^2$

(d) $3t^2 - 7t - 6$

(e) $9s^2 - 16$

(f) $u^2 + 2uv + v^2$

(g) $6x^2 - x - 7$

(h) $3y^2 - 3$

(i) $u^2 + 5u - 14$

(j) $y^2 - 10y + 25$

(k) $1 - 9t^2$

(l) $s^2 + 3s + 2$

(m) $8y^2 + 8$

(n) $2x^2 + 5x + 2$

(o) $8m^2 - 8$

(p) $4y^2 - 4y + 1$

(q) $ax^2 + 10ax + 25a$

(r) $3r^2 - 48$

(s) $4x^2 - 20x$

(t) $5m^2 - 20$

(u) $3ax^2 - 15ax + 18a$

(v) $3t^2 + 8t + 5$

(w) $a^2x - b^2x$

(x) $2k^2 - 2k - 12$

(y) $t^4 - 1$

(z) $5y^2 + 18y - 72$

13-7. Polynomials over the Real Numbers.

When we stated that an expression such as

$$x^2 - 5$$

could not be factored, this meant that $x^2 - 5$ could not be expressed as the product of two polynomials over the integers. However, if we form the product

$$(x + \sqrt{5})(x - \sqrt{5}),$$

we see that this is equal to $x^2 - 5$. Check this by multiplication! In other words, $x^2 - 5$ does have polynomial factors. But these factors are polynomials over the real numbers, not polynomials over the integers.

In the beginning of the chapter we stressed the fact that the problem of factoring involved finding factors which were expressions of the same type as the particular expression being factored. How does this idea relate to the example considered above? The answer is this: The polynomial

$$x^2 - 5$$

is a polynomial over the integers. However, since 5 is also an element in the set of real numbers, we could regard $x^2 - 5$ as a polynomial over the reals as well. We thus have two possible situations.

(1) If we are working with the basic set of integers as coefficients, then we say that $x^2 - 5$ can not be factored.

(2) If we are working in the larger set of real numbers, then we say that $x^2 - 5$, as a polynomial over the reals, can be factored.

In short, the question of whether or not a given polynomial can be factored depends on the basic set of numbers from which the coefficients may be chosen.

It should be noted that the polynomial

$$x^2 - 9$$

can be factored as a polynomial over the integers and also as a polynomial over the reals. In both cases the factored form is

$$(x + 3)(x - 3).$$

The expression

$$9x^2 - 2$$

cannot be factored over the integers. It can be factored over the reals. The factored form is

$$(3x + \sqrt{2})(3x - \sqrt{2}).$$

Check this by multiplication!

Check Your Reading

1. Is $x^2 - 5$ a polynomial over the integers? Can it be regarded as a polynomial over the real numbers?
2. Does $x^2 - 5$ have factors if it is considered to be a polynomial over the integers? Does it have factors if it is considered to be a polynomial over the real numbers?
3. Factor $x^2 - 9$ over the integers. Factor it over the reals.
4. Can $9x^2 - 2$ be factored over the integers? Factor it over the reals.

Oral Exercises 13-7

Factor the following expressions over the real numbers.

1. $x^2 - 7$
2. $y^2 - 11$
3. $9m^2 - 5$
4. $4z^2 - 3$
5. $7k^2 - n^2$
6. $9x^2 - 5y^2$
7. $a^2x^2 - 3$
8. $25a^2y^2 - 7b^2$
9. $2a^2 - 3$
10. $a^2 - c \quad , \quad c > 0$

Problem Set 13-7

Factor the following expressions over the real numbers.

1. $m^2 - 5$
2. $t^2 - 16$
3. $4x^2 - 7$
4. $4y^2 - 9$
5. $a^2b^2 - 3$
6. $z^2 - 9a^2$
7. $k^2 - 7b^2$
8. $36u^2 - 5v^2$
9. $3x^2 - 2$
10. $r^2 - c \; , \; c > 0$

13-8. Polynomials over the Rationals.

Let us now consider the polynomial

$$\frac{1}{3}x^2 + \frac{4}{3}x + \frac{4}{3} .$$

This is a polynomial over the rational numbers. It can be factored, since rational numbers are allowed as factors. In this example we want to "factor out" the monomial $\frac{1}{3}$ and obtain

$$\frac{1}{3}(x^2 + 4x + 4).$$

Do you see that in this way we have obtained a factor on the right which is a polynomial over the integers as well as over the rationals? In this particular case it is also a perfect square. The complete factored form is

$$\frac{1}{3}(x + 2)(x + 2).$$

Let's now look at another polynomial over the rationals.

$$\frac{2}{3}y^2 - 6$$

This can be written as

$$\frac{2}{3}y^2 - \frac{2}{3}(9).$$

If we factor out $\frac{2}{3}$, we get

$$\frac{2}{3}(y^2 - 9).$$

This result should also be checked by multiplication. Again the factor on the right is a polynomial over the integers. In this instance we have the difference of two squares. Thus further factoring is possible. The final form is

$$\frac{2}{3}(y + 3)(y - 3).$$

In each of the above examples of polynomials over the rationals we have factored out a rational number thereby obtaining a polynomial over the integers as the other factor. Is this always possible? The following example will provide a clue to the answer.

If we consider the following expression:

$$\frac{1}{3}x^2 + \frac{1}{4}x + \frac{1}{6} ,$$

we see that it can be written as

$$\frac{4}{12}x^2 + \frac{3}{12}x + \frac{2}{12} .$$

Can you find a rational number which, when factored out, will leave a polynomial over the integers? Our expression can be written as

$$\frac{1}{12}(4x^2 + 3x + 2)$$

Although the factor on the right cannot be factored any further, we do see that it is a polynomial over the integers.

It should be clear that the rational number which was factored out in the above example is related to the common denominator of the fractions in the original polynomial. In fact it is the reciprocal of this common denominator. Do you see, then, why it is reasonable to conclude that

> any polynomial over the rationals can be written as the product of a rational number and a polynomial over the integers.

Check Your Reading

1. Complete the sentence: $\frac{1}{3}x^2 + \frac{4}{3}x + \frac{4}{3}$ is a polynomial over the ____________________.
2. What is the first step in factoring $\frac{2}{3}y^2 - 6$ over the rationals?
3. Is it always possible to "factor out" a rational monomial from a polynomial over the rationals so that the other factor is a polynomial over the integers?

Oral Exercises 13-8

Express each of the following polynomials over the rationals as a product of a rational number and a polynomial over the integers.

1. $\frac{1}{2}x^2 - \frac{9}{2}$
2. $\frac{1}{3}t^2 + \frac{2}{3}t + \frac{1}{3}$
3. $\frac{1}{5}x^2 - \frac{4}{5}$
4. $\frac{2}{3}y^2 + \frac{4}{3}$
5. $\frac{1}{4}m^2 + \frac{1}{2}$
6. $\frac{3}{2}s^2 - \frac{1}{6}$
7. $\frac{1}{2}y^2 - 2$
8. $\frac{1}{6}m^2 + \frac{1}{2}m + 1$
9. $\frac{1}{3}y^2 - \frac{1}{4}$
10. $3 - \frac{2}{3}x^2$

Problem Set 13-8

Express each of the following polynomials over the rationals as a product of a rational number and prime polynomial(s) over the integers.

1. $\frac{1}{2}t^2 - \frac{9}{2}$
2. $\frac{1}{3}y^2 + \frac{4}{3}y + \frac{4}{3}$
3. $\frac{1}{5}z^2 + \frac{4}{5}$
4. $\frac{1}{12}x^2 - \frac{1}{3}$
5. $\frac{1}{8}x^2 - \frac{1}{8}x - \frac{3}{4}$
6. $\frac{1}{3}x^2 + \frac{2}{3}x$
7. $\frac{1}{2}m^2 - 2$
8. $\frac{1}{6}a^2 + \frac{1}{3}ab + \frac{1}{6}b^2$
9. $\frac{1}{4}x^2 - 4$
10. $5 - \frac{1}{5}t^2$

13-9. Truth Sets of Polynomial Equations.

A polynomial equation is an equation in which each side is a polynomial. An open sentence of this type can be put in a form in which the left side is a polynomial and the right side is 0. If we can factor the polynomial on the left side, it will be very helpful in finding the truth set of the equation.

Example: Find the truth set of the equation

$$x^2 - x - 6 = 0.$$

Our first job is to factor the polynomial, if possible. Once this is done we can write the equation in factored form as follows:

$$(x + 2)(x - 3) = 0.$$

You should check to see that these are the correct factors. If x is a real number, then each of these factors can be thought of as a real number.

Our equation, then, is of the form

$$a \cdot b = 0$$

where a and b are each real numbers. In Chapter 4 we learned that if any two real numbers are such that their product is equal to zero, then either one or the other of the numbers must be zero. We also learned that if either a or b is zero, then the product $a \cdot b$ must be zero.

We can now make an important statement about the equation

$$(x + 2)(x - 3) = 0.$$

We can say that the sentence is true if

$$(x + 2) = 0.$$

It is also true if

$$(x - 3) = 0.$$

The sentence is not true under any other conditions.

We can see then that our problem has been changed to that of finding the truth set of the compound sentence

$$(x + 2) = 0 \qquad \text{or} \qquad (x - 3) = 0.$$

We see by the addition property of equality that the truth set of the left clause is $\{-2\}$ and the truth set of the right clause is $\{3\}$. From this we learn that the truth set of our polynomial equation is

$\{-2, 3\}$. Check this in the original sentence.

As a second example, let us try to find the truth set of the sentence

$$x^2 = 4x + 5.$$

To put this sentence into a form in which one side is zero we must first apply the addition property of equality. We add

$$(-4x) \quad \text{and} \quad (-5)$$

to both sides. Our sentence then takes the form

$$x^2 - 4x - 5 = 0.$$

We must now write the polynomial on the left side in factored form. Can you see that our equation becomes

$$(x - 5)(x + 1) = 0\ ?$$

Its truth set is the same as that of the compound sentence

$$x - 5 = 0 \qquad \text{or} \qquad x + 1 = 0.$$

The truth set of this compound sentence, and hence that of the original sentence $x^2 = 4x + 5$, is

$\{5, -1\}$. (Check these values in the original sentence.)

Now suppose we were asked to solve the equation

$$x^2 - 4x - 9 = 0.$$

Notice that we have used the expression "solve the equation" to mean the same thing as "find the truth set of the sentence". We shall do this often throughout the book.

When we try to factor the polynomial

$$x^2 - 4x - 9 = 0,$$

we find that there are no two integers whose product is (-9) and whose sum is (-4). Therefore, we cannot factor this polynomial over the integers. This does not necessarily mean that the truth set of the sentence is the empty set. It means that we cannot find the truth set by this method of factoring.

Check Your Reading

1. What do we call an equation such as $x^2 - x - 6 = 0$?
2. If $a \cdot b = 0$, what do you know about either a or b?
3. If $x + 2 = 0$, what is the value of x?
4. If $x - 3 = 0$, what is the value of x?
5. What property is used to find the value of x in questions 3 and 4?
6. How can you write $x^2 = 4x + 5$ so that it is a polynomial equation in which one side is zero?

Oral Exercises 13-9

Give the truth set of each of the following:

1. $(x + 3)(x + 2) = 0$
2. $(b - 2)(b + 4) = 0$
3. $(m - 3)(m + 3) = 0$
4. $(c - 5)(c - 2) = 0$
5. $(a - 9)(a + 1) = 0$
6. $a(a - 1) = 0$

Problem Set 13-9

1. Find the truth set of each of the following polynomial equations.
   - (a) $x^2 + 5x + 4 = 0$
   - (b) $x^2 - 5x + 6 = 0$
   - (c) $a^2 + 3a + 2 = 0$
   - (d) $b^2 - 8b - 9 = 0$
   - (e) $w^2 - 5w - 14 = 0$
   - (f) $t^2 + 2t - 15 = 0$
   - (g) $b^2 + 17b + 72 = 0$
   - (h) $a^2 - 2a + 1 = 0$
2. Write each of the following as a polynomial equation with one side zero. Then find the truth set of each equation.
   - (a) $t^2 + 2t = 15$
   - (b) $x^2 + 11x = -18$
   - (c) $y^2 + 3 = 4y$
   - (d) $b^2 + 5 = 6b$
   - (e) $m^2 = 66 + 5m$
   - (f) $a^2 = 13a + 30$
   - (g) $x^2 = 5x - 6$
   - *(h) $18 + 7x - x^2 = 0$

Problem Set 13-9
(continued)

*(i) $3x^2 - 11x + 30 = 2x^2 - 2x + 10$

*(j) $y^2 - 49 = 0$

*(k) $x^2 = -8x - 36$

3. (a) The square of a number is 7 greater than 6 times the number. What is the number?

   (b) The length of a rectangle is 5 inches more than its width. Its area is 84 square inches. Find its width.

   (c) The square of a number is 9 less than 10 times the number. What is the number?

   (d) If one number is 8 less than another and their product is 84, find the numbers.

   (e) The product of two consecutive odd numbers is 15 more than 4 times the smaller number. What are the numbers?

   *(f) Find the dimensions of a rectangle whose perimeter is 28 feet and whose area is 24 square feet. Hint: If the perimeter is 28 feet then the sum of the length and width is 14 feet; if ℓ represents the number of feet in the measurement of the length then $(14 - \ell)$ represents the number of feet in the measurement of the width of the rectangle.

   *(g) A rectangle and a square have equal areas. The length of the rectangle is 4 feet more than the number of feet in a side of the square. The width of the rectangle is 3 feet less than the number of feet in a side of the square.

      (1) Find the number of feet in a side of the square.

      (2) Find the dimensions of the rectangle.

Summary

In this chapter we have been concerned with certain types of phrases called polynomials. In forming our definition of a polynomial we began by considering the set of integers:

$I = \{\ldots (-3), (-2), (-1), 0, 1, 2, 3, \ldots\}$,

and certain variables. A polynomial was then defined as

> "any numeral for an element in the set, or any variable, or any expression which indicates operations of addition, subtraction, multiplication, or taking opposites of any elements of the set and the variables."

Thus, $x^2 + 3x - 7$ is a polynomial; $\frac{4}{x + 3}$ is not.

In choosing the original set of coefficients as the set of integers we were really forming what should have been called polynomials over the integers. However, for convenience we agreed to refer to this type simply as polynomials. If the original set of coefficients is to be the rational, or the real numbers, then in the future we will refer to the latter types of polynomials as polynomials over the rationals, or polynomials over the reals.

Factoring has been discussed as a study of the ways in which certain types of polynomials can be written as products of other polynomials of the same type. The two principal factoring methods examined were:

(1) Common monomial factor.

Example: $4st + 8st^2 - 28s^2t^3 = 4st(1 + 2t - 7st^2)$

(2) The Quadratic Polynomial.

Examples: $x^2 + 8x + 15 = (x + 5)(x + 3)$

$6x^2 - 7x - 3 = (3x + 1)(2x - 3)$

Two special types of quadratic polynomials were considered which, because of their particular formation, could be factored by inspection. These were

(2a) The difference of two squares.

Example: $25x^2 - 9 = (5x + 3)(5x - 3)$

(2b) Perfect Squares.

Example: $4x^2 - 12x + 9 = (2x - 3)(2x - 3)$

Consideration has been given to factoring polynomials over the reals.

Example: $x^2 - 7 = (x + \sqrt{7})(x - \sqrt{7})$

With respect to polynomials over the rationals the property was established that

> any polynomial over the rationals can be written as the product of a rational number and a polynomial over the integers.

Example: $\frac{1}{5}x^2 - 2x + 5 = \frac{1}{5}(x^2 - 10x + 25)$

$= \frac{1}{5}(x - 5)(x - 5)$

We have also studied the solution of polynomial equations by means of factoring.

Example: $x^2 - 7x + 12 = 0$

This may be written as $(x - 3)(x - 4) = 0$.
From this we obtain the compound sentence

$x - 3 = 0$ or $x - 4 = 0$,

which becomes $x = 3$ or $x = 4$.

We conclude that the truth set is $\{3, 4\}$.

Review Problem Set

1. Find a common name for each of the following indicated sums:

   (a) 12 + 7

   (b) 22 + (-6)

   (c) (-8) + 30

   (d) (-26) + 12

   (e) 10 + (-21)

   (f) (-9) + (-17)

   (g) (-4) + 14 + (-6) + (-4)

   (h) 26 + (-10) + 4 + (-9)

2. Use the distributive property to collect terms in each of the following:

   (a) 6a + 5a

   (b) 8b + (-3b)

   (c) (-6a) + 9a

   (d) -5c + 5c

   (e) 3t - 6t

   (f) (-4a) + (-8a)

   (g) 7d - 3d - 2d

   (h) 10a - 4a + 3a - 14a

   (i) 5c + 4b - 3c - 6b

   (j) -6m + 5n + 2m

   (k) 2x - 3y + 4z

3. Find a common name for each of the following indicated products:

   (a) (10)(2)

   (b) (-3)(9)

   (c) (8)(-4)

   (d) (-5)(-6)

   (e) (-2)(-3)(5)

   (f) (-7)(2)(-1)

   (g) (-8)(-3)(-1)

   (h) (-5)(-7)(12)(0)

   (i) (2a)(-4)

   (j) (-3m)(6)

   (k) (-6m)(-3)

   (l) (1.25a)(-3)

   (m) $(\frac{7}{8})(-3b)$

   (n) $(.9a)(-4)(\frac{7}{8})(-\frac{6}{6})(0)$

Review Problem Set
(continued)

4. Use the distributive property to simplify each of the following:

(a) $a(b + c)$
(b) $a\big(b + (-c)\big)$
(c) $m\big((-a) + (-b)\big)$
(d) $2m(n + 3p)$
(e) $(-6m)(m - 2n)$
(f) $(-3a - 5b)(-3c)$
(g) $(a + b + c)(-1)$
(h) $(-1)(4a - 3b + 2m - 7)$
(i) $(-4x - 3y + 3x + x + 3y)(-2mn)$
(j) $(a + 3)(a + 2)$
(k) $(m - 4)(m - 3)$
(l) $(a + 6)(a - 4)$
(m) $(2a + 3)(3a + 4)$
(n) $(4b + 1)(2b - 1)$
(o) $(3c - 2)(5c - 6)$
(p) $(4m - 3n)(m - 5n)$
(q) $(4m - n)(4m + n)$
(r) $(a + b)(c + d)$

5. Find a common name for each of the following:

(a) $|6| - |2|$
(b) $|-9| - |-4|$
(c) $|7| - |10|$
(d) $|-3| - |-8|$
(e) $|5 - 1|$
(f) $|-4 - 7|$

6. Simplify the following expressions. (Collect terms if possible.)

(a) $(5x - 6) + (2x + 1)$
(b) $3y + 7 + 2y - 80$
(c) $x^2 + 6x + 6 + 3x^2 - x - 8$
(d) $-3n - 7 - n + 8 + n + 2$
(e) $(2a + b) - (a - b)$
(f) $(6x - 3y) - \big((-3)x + 2y\big)$
(g) $(3b + 2c - 4) + \big((-3)b + 4c\big)$
(h) $(5m + 7n - 8) - (3m - 2n + 6)$
(i) $(5x - y - 2z) - \big((-3)x + 6y + 2z\big)$
(j) $(2r - 3s + 6t) - \big((-5)r + 7s + (-2)t\big)$

Review Problem Set
(continued)

7. Factor the following into prime factors.

(a) $6a + 12b$

(b) $c + 3c$

(c) $m^2 + 2mb + m$

(d) $-3a + 6b - 12c$

(e) $2ab + 8ax - 10ay$

(f) $x^2 + 7x + 3x + 21$

(g) $6mr + 14r + 15ms + 35s$

(h) $ac - ad + bc - bd$

(i) $15m^2 + 21mr - 20mn - 28nr$

(j) $15ac - 20ad - 3bc + 4bd$

(k) $b^2 + 11b + 30$

(l) $a^2 - 3a - 70$

(m) $m^2 + 7m - 8$

(n) $3c^2 - 11cd + 10d^2$

(o) $2ab^2 - 5abc + 2ac^2$

(p) $abm^2 - 64ab$

(q) $12c^2t - 75d^2t$

(r) $x^2 - 16x + 64$

(s) $9x^2 + 60x + 100$

(t) $4x^2 - 20ax + 25a^2$

(u) $5 - 4s^2$

8. Simplify each of the following:

(a) $\frac{1}{10} + \frac{4}{15}$

(b) $\frac{a}{2} + \frac{b}{4}$

(c) $\frac{m}{18} + \frac{n}{15} + \frac{3}{10}$

(d) $\frac{1}{10a} - \frac{6}{2a}$

(e) $(\frac{3}{8})(-\frac{4}{9})$

(f) $(\frac{m}{9})(\frac{m}{4})$

(g) $\frac{a^2b}{m^2} \cdot \frac{a^3b^4}{n}$

(h) $\frac{x^4y^2z^3}{a^2b^3} \cdot \frac{ab^6}{x^8z}$

(i) $-\left((\frac{1}{a})am^4\right)$

(j) $-\frac{10m^{10}n^2}{7xn} \cdot \frac{35x^5}{18m^{12}n}$

(k) $\dfrac{\frac{7}{8}}{\frac{2}{3}}$

(l) $\dfrac{\frac{10a^2}{3a}}{\frac{4m}{6b}}$

Review Problem Set
(continued)

(m) $$\frac{-\dfrac{7m^4n^3}{5x^2}}{-\dfrac{3m}{15ax}}$$

(n) $$\frac{\dfrac{6mx}{7c^2d}}{-\dfrac{12mx}{14c^2d}}$$

9. Find the truth set of each of the following open sentences:

   (a) $2x - 7 = 23$

   (b) $8(3x - 7) = 24$

   (c) $3(x - 2) = 2(x - 1) + 15$

   (d) $10 - (4 + 2x) = 12 + 2(2x + 3)$

   (e) $x^2 - 81 = 0$

   (f) $3x^2 - 2x - 1 = 0$

   (g) $2(x^2 + 2) = 9x$

   (h) $x^2 - 3x + 4 = 2(x + 2)$

   (i) $(-3)y + 27 > (-5)y + 13$

   (j) $|y - 3| < 4$

   (k) $|y| < (-10)$

10. Translate each of the following into open sentences and find their truth sets:

   (a) The perimeter of a rectangle is 80 ft. Its length is 8 ft. greater than its width. Find its length and width.

   (b) Ann is three times as old as Jerry. In 12 years Ann will be only twice as old as Jerry. Find their present ages.

   (c) The number of girls in an algebra class of 18 pupils is 2 less than the number of boys. Find the number of each.

Review Problem Set
(continued)

(d) The sum of two numbers is 15 and the sum of their squares is 137. Find the numbers.

(e) The altitude of a triangle is 3 inches less than its base. Its area is 14 square inches. What is the length of its base?

Chapter 14

RATIONAL EXPRESSIONS

14-1. Polynomials and Integers.

In the previous chapter, it was seen that expressions such as

$$2x + 3 \quad \text{and} \quad x^2 - 2x + 5$$

are called polynomials over the integers. Any two such polynomials can be added, subtracted, or multiplied. Furthermore, the result in each case is another polynomial over the integers. For example, using the two polynomials above, the sentences

$$(2x + 3) + (x^2 - 2x + 5) = x^2 + 8,$$

$$(2x + 3) - (x^2 - 2x + 5) = -x^2 + 4x - 2, \text{ and}$$

$$(2x + 3)(x^2 - 2x + 5) = 2x^3 - x^2 + 4x + 15,$$

illustrate the fact that the sum of these polynomials, their difference, and their product are also polynomials over the integers.

We can say then that the set of all polynomials over the integers is closed under addition, subtraction, and multiplication. What about division? Are these polynomials closed under division? Using again the two polynomials at the beginning of this section, their quotient might be indicated in this way:

$$\frac{x^2 - 2x + 5}{2x + 3}.$$

Is the above expression a polynomial? Obviously, it is not, since the definition of polynomial (see Chapter 13) does not permit the operation of division. Thus, dividing one polynomial by another does not always produce another polynomial. That is, the set of polynomials over the integers is not closed under division.

Earlier, in our study of the real numbers, we met a set of numbers which is closed under addition, subtraction, and multiplication, but not under division. This was the set of integers,

$$\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}.$$

Do you recall that the sum of two integers, the difference of two integers, and the product of two integers are also integers? However, the quotient of two integers is not always an integer---for example, $\frac{3}{2}$ is not an integer.

In this way, then, the set of integers is like the set of polynomials over the integers. Both sets are closed under addition, subtraction, and multiplication, but not under division. Although we do not discuss them at this time, there are other ways in which the "behavior" of the integers is similar to the behavior of the polynomials over the integers. We shall make further use of this similarity later in the chapter.

Check Your Reading

1. Using the polynomials, "$2x + 3$" and "$x^2 - 2x + 5$," which of the following are polynomials over the integers?

   $(2x + 3) + (x^2 - 2x + 5)$, $(2x + 3) - (x^2 - 2x + 5)$,

   $(2x + 3)(x^2 - 2x + 5)$, $\frac{x^2 - 2x + 5}{2x + 3}$

2. Under what operations is the set of polynomials over the integers closed?

3. 3 and 2 are integers. Which of the following are integers: $3 + 2$, $3 - 2$, $(3)(2)$, $\frac{3}{2}$?

4. Under what operations is the set of integers closed?

Problem Set 14-1

1. Simplify each of the following expressions. Is the resulting expression a polynomial in each case?

   (a) $(3x + 2) + (2x - 4)$
   (b) $(7x - 4) - (2x + 7)$
   (c) $(4a + 2) + (a + 3) + (7a - 1)$
   (d) $(a + 1)(a - 1)$

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Problem Set 14-1
(continued)

(e) $(a^2 + 3a - 1) + (2a^2 + 2)$
(f) $(3a^2 - 2a) - (2a - 1)$
(g) $(2x + 7)(a - 1)$
(h) $(x + 5)^2$
(i) $(1 + 2y + y^2) - (y^2 + 2y + 1)$
(j) $(5x^3 - 3x^2 + 2x) - (2x^3 - 3x + 7)$

2. Find the truth set of each of the following sentences by simplifying polynomial expressions where necessary.

(a) $(3x + 2 - x^2) + (2x + x^2 - 1) = 0$
(b) $(3a - 7) - (2a + 4) = a - 5$
(c) $(4x + 1)(x - 2) - 4x^2 = 0$

3. Two triangles each have a base of length 4 inches. The height of one is $\frac{3}{2}$ the height of the other. The total area of the two triangles is 15 square inches. What is the height of each triangle?

14-2. Quotients of Polynomials.

As observed in the previous section, the quotient of two polynomials is not always a polynomial. However, an expression such as

$$\frac{x^2 - 2x + 5}{2x + 3},$$

which is the indicated quotient of two polynomials, is an important kind of expression. It is important that we be able to work with such expressions, to add them and to multiply them. We shall turn our attention to these operations in the following sections. First, however, the problem of restricting the domain of a variable in such expressions must be considered.

An indicated quotient in which the denominator is 0----for example, $\frac{6}{0}$ ---does not name a real number. Therefore, if we are to work with indicated quotients of polynomials, we must be careful to exclude values of the variable which make the denominator zero. We shall then be able to say that the indicated quotient of two polynomials represents a real number.

Earlier, we have seen examples in which the domain of a variable had to be restricted. Below are some additional examples which may help you to work the problems that follow.

Example 1. For what values of x does "$\frac{3}{x - 2}$" represent a real number?

"$\frac{3}{x - 2}$". names a real number provided that the denominator is not 0.

If $x = 2$, the denominator is 0. No other value of x makes the denominator 0. Therefore, in working with this expression, it must be stated that x is not 2.

It could also be said that the domain of x includes all real numbers except the number 2.

Example 2. What restriction must be placed on the domain of y in the expression

$$\frac{3y + 2}{(2y - 5)(y + 7)} ?$$

If $y = \frac{5}{2}$, $(2y - 5)$ is 0. If $y = -7$, $(y + 7)$ is 0. In either case, the denominator of the above expression is 0. (Why?)

Therefore, the domain of y includes all real numbers except $\frac{5}{2}$ and -7.

Example 3. What is the domain of n in the expression

$$\frac{4n + 1}{n^2 + 3n + 2} ?$$

$$\frac{4n + 1}{n^2 + 3n + 2} = \frac{4n + 1}{(n + 2)(n + 1)} .$$

Therefore, it must be stated that $n \neq -2$ and $n \neq -1$.

Example 4. What restriction must be placed on the domain of z in "$\frac{2z}{z^2 + 3}$" ?

If $z = 0$, $z^2 = 0$ and the denominator "$z^2 + 3$" is 3. For all other values of z, z^2 is positive, and the sum of a positive number and 3 is not zero. Therefore, there is no value of z that makes the denominator zero.

The domain of z is the set of all real numbers.

Check Your Reading

1. Is "$\frac{6}{0}$" the name of a real number?
2. Does "$\frac{3}{x - 2}$" represent a real number for every value of x?
3. What restriction must be placed on the domain of x in the expression "$\frac{3}{x - 2}$" ?
4. What is the domain of y in the expression "$\frac{3y + 2}{(2y + 5)(y + 7)}$"?
5. Does "$\frac{2z}{z^2 + 3}$" represent a real number for every value of z?

Oral Exercises 14-2

1. What values must be excluded from the domains of the variables in the indicated quotient of two polynomials?
2. For what values of x does $\frac{3}{x + 1}$ name a real number?
3. Explain why the domain of x for the expression $\frac{1}{x} + 1$ is not the set of all real numbers.
4. Tell what numbers, if any, must be excluded from the domain of the variables in each of the following expressions.

(a) $3x + 1$ (e) $\frac{5}{0 + x}$

(b) $\frac{1}{x} + \frac{2}{x}$ (f) $\frac{a^2 + 2ab}{2b + 1}$

(c) $x^2 + 2mx + m^2$ (g) $|a|$

(d) $\sqrt{x}$ (h) $\frac{1}{4}(x + 7)^2$

Oral Exercises 14-2
(continued)

(i) $\frac{1}{x + 2}$

(j) $\frac{1}{a - 1} + \frac{2}{a + 2}$

(k) $(a + b)^2$

(l) $\frac{x - 4}{3x - 5}$

(m) $\sqrt{x + 1}$

(n) $\frac{5(x + 7)^2}{x(3x - 2x - x)}$

(o) $\sqrt{(x + 1)^2}$

Problem Set 14-2

State the restrictions, if any, that must be placed on the domains of the variables in each of the following expressions or sentences.

1. $x^2 + 2x$
2. $\frac{2a}{a - 1}$
3. $\frac{3x}{x + 3}$
4. $\frac{a + 5}{b + 5}$
5. $\frac{3x^2 + 2x + 1}{2x - 5}$
6. $\sqrt{x^2 + y^2}$
7. $\frac{1}{9xy} + \frac{5}{y}$
8. $|a| = b$
9. $\frac{3x}{(7x + 4)^2}$
10. $\frac{2x + 7}{(x + 3)(x - 3)-(x^2 - 9)}$
11. $\frac{x - 2}{x^2 + 5x + 6}$
12. $\frac{n}{n^2 + 5}$
13. $\frac{n}{n^2 + n}$
14. $\frac{a - 3}{(a + 7)(a - 3)}$
15. $\frac{7}{x^2 - 2}$

14-3. Multiplying Quotients of Polynomials

In working with real numbers, it turned out that the basic operations are addition and multiplication. Subtraction and division can be defined in terms of addition and multiplication.

Since, with necessary restrictions on the variable, quotients of polynomials represent real numbers, our principal concern in working with such expressions will be with the operations of multiplication and addition. In this section, we consider multiplication. In section 14-4, addition is considered.

Let us begin by looking at the expression

$$\frac{x}{x - 5} \cdot \frac{2x - 3}{4} ,$$

which is the indicated product of two "polynomial quotients." (Remember that the definition of polynomial allows us to consider "x" and "4" as polynomials.) First, what restriction must be placed on the domain of x? $\underline{x}$ cannot be $\underline{5}$. However, if x is not 5, then both "$\frac{x}{x - 5}$" and "$\frac{2x - 3}{4}$" represent real numbers, and the sentence

For all real numbers a, b, c, d, $b \neq 0, d \neq 0$,

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} ,$$

may be applied as follows:

$$\frac{x}{x - 5} \cdot \frac{2x - 3}{4} = \frac{x(2x - 3)}{(x - 5)4} .$$

$$= \frac{2x^2 - 3x}{4x - 20} .$$

The result itself is the quotient of two polynomials.

Thus, multiplying two quotients of polynomials is not really a new idea at all. In fact, we have worked with such expressions in previous chapters. Below are two other examples, illustrating the kind of problems you are asked to consider in the problem set.

Example 1. Simplify $\frac{y + 3}{4y} \cdot \frac{2y}{y - 4}$. This expression is the indicated product of two polynomial quotients.

The domain of y must be restricted so that the denominators "$4y$" and "$y - 4$" are not zero. This means that y cannot be 0, and y cannot be 4.

If $y \neq 0$ and $y \neq 4$,

$$\frac{y + 3}{4y} \cdot \frac{2y}{y - 4} = \frac{(y + 3)(2y)}{(4y)(y - 4)}$$

$$= \frac{(2)(y)(y + 3)}{(2)(2)(y)(y - 4)}$$

$$= \frac{(2)(y)}{(2)(y)} \cdot \frac{(y + 3)}{y(y - 4)}$$

"$\frac{2y}{2y}$" is a name for 1, if $y \neq 0$.

$$= (1) \cdot \frac{y + 3}{y(y - 4)}$$

$$= \frac{y + 3}{y(y - 4)} .$$

This result may also be written "$\frac{y + 3}{y^2 - 4y}$", which is itself the quotient of two polynomials.

Example 2. Simplify $\frac{3}{2x - 1} \cdot \frac{6x - 3}{3x}$.

$$\frac{3}{2x - 1} \cdot \frac{6x - 3}{3x} = \frac{3}{2x - 1} \cdot \frac{3(2x - 1)}{3x}$$

Notice that the phrase "$6x - 3$" in the original expression is not prime. In the first step, it is expressed as the product of prime factors, $3(2x - 1)$. This is often a helpful step in simplifying.

If $x \neq 0$, $x \neq \frac{1}{2}$,

$$\frac{3}{2x - 1} \cdot \frac{3(2x - 1)}{3x} = \frac{3\big(3(2x - 1)\big)}{(2x - 1)(3x)}$$

$$= \frac{3\big(3(2x - 1)\big)}{x\big(3(2x - 1)\big)}$$

$$= \frac{3}{x} \cdot \frac{3(2x - 1)}{3(2x - 1)}$$ "$\frac{3(2x - 1)}{3(2x - 1)}$" is a name for 1, if $x \neq \frac{1}{2}$.

$$= \frac{3}{x} \cdot (1)$$

$$= \frac{3}{x}.$$ Again the result is itself the quotient of two polynomials.

Check Your Reading

1. Complete the following sentence: For any real numbers a, b, c, d, $b \neq 0$, $d \neq 0$, $\frac{a}{b} \cdot \frac{c}{d} =$ ______.
2. Complete the following sentence: If $x \neq 5$, $\frac{x}{x - 5} \cdot \frac{2x - 3}{4} =$ ______.
3. If $x \neq \frac{1}{2}$, "$\frac{3(2x - 1)}{3(2x - 1)}$" is a name for what number?
4. Complete this sentence: If $x \neq \frac{1}{2}$, $x \neq 0$, "$\frac{3}{x} \cdot \frac{3(2x - 1)}{3(2x - 1)}$" may be simplified to ______.
5. If two quotients of polynomials are multiplied, is the result also a quotient of polynomials?

Oral Exercises 14-3a

1. For what values of x is it true that $\frac{2x - 1}{2x - 1} = 1$?
2. State the restrictions on the domains of the variables and then, if possible, simplify each of the following expressions.

(a) $\frac{1}{x} \cdot \frac{2}{3x}$ (b) $\frac{k}{3} \cdot \frac{3}{k}$

Oral Exercises 14-3a
(continued)

(c) $\frac{a + 2}{3a} \cdot \frac{a}{a + 2}$

(d) $\frac{3mx}{m^2} \cdot \frac{2x}{x^2}$

(e) $\frac{m}{2} \cdot \frac{m + 2}{3}$

(f) $\frac{x + 2}{2x} \cdot \frac{x}{x + 2}$

(g) $\frac{x - 2}{4} \cdot \frac{x + 2}{x}$

(h) $(k + 3) \cdot \frac{1}{k + 3}$

Problem Set 14-3a

Perform each of the following indicated multiplications, simplifying where possible. State the restrictions on the domains of the variables.

1. $\frac{2}{x} \cdot \frac{11}{x}$
2. $\frac{ab}{c} \cdot \frac{a}{bc}$
3. $\frac{3(x - 1)}{x + 2} \cdot \frac{5(x + 4)}{x - 1}$
4. $\frac{3a + 36}{2b} \cdot \frac{b^2}{a + 12}$
5. $\frac{5m}{2m^2 + m} \cdot \frac{4m + 2}{5}$
6. $\frac{a - b}{a + b} \cdot \frac{2a + 2b}{3a - 3b}$
7. $\frac{a - b}{3} \cdot \frac{2}{b - a}$
8. $\frac{3a^2bc}{2a} \cdot \frac{7b^2c^2}{3a^2} \cdot \frac{a^3}{14c^3}$
9. $\frac{ax^2 + a}{b} \cdot \frac{b^2}{x^2 + 1}$
10. $\frac{4k + 12}{4k - 4} \cdot \frac{(k - 1)}{(k + 3)}$
11. $\frac{5x + 15}{6x - 18} \cdot \frac{2x + 6}{x^2 - 3x}$
12. $\frac{(x + 1)(x - 1)}{3x + 3} \cdot \frac{x + 1}{x - 1}$
13. $\frac{(x + 1)x + (x + 1)2}{x + 2} \cdot \frac{5}{x + 1}$

No matter how "complicated" the polynomials may be, the simplification of an indicated product of two quotients of polynomials can be accomplished by the following:

(1) Restrict the domain of the variables so that the indicated quotients represent real numbers;

(2) Apply the sentence, "For all real numbers a, b, c, d, $b \neq 0$, $d \neq 0$, $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$;"

(3) If possible, use the multiplication property of one to simplify the result.

Example. Simplify $\frac{x^2 + 3x - 10}{x^2 + 3x} \cdot \frac{2x + 6}{x^2 - 4x + 4}$.

Four polynomials are present in the expression above. As mentioned earlier, it is often helpful to express each polynomial as a product of prime polynomials, in this case with the following result:

$$\frac{(x + 5)(x - 2)}{x(x + 3)} \cdot \frac{2(x + 3)}{(x - 2)(x - 2)}$$

If $x \neq 0$, $x \neq -3$, and $x \neq 2$,

$$\begin{aligned}\frac{(x + 5)(x - 2)}{x(x + 3)} \cdot \frac{2(x + 3)}{(x - 2)(x - 2)} &= \frac{(x + 5)(x - 2)(2)(x + 3)}{x(x + 3)(x - 2)(x - 2)} \\ &= \frac{(x - 2)(x + 3)}{(x - 2)(x + 3)} \cdot \frac{(2)(x + 5)}{(x)(x - 2)} \\ &= (1) \cdot \frac{2(x + 5)}{x(x - 2)} \\ &= \frac{2x + 10}{x^2 - 2x}.\end{aligned}$$

Again, the result is a single indicated quotient of polynomials.

Problem Set 14-3b

Perform each of the following indicated multiplications, simplifying where possible. State the restrictions on the domains of the variables.

1. $\frac{3x - 6}{x^2 + 2x} \cdot \frac{5x + 10}{15}$

2. $\frac{m + 5}{m - 5} \cdot \frac{m^2 - 25}{3m}$

3. $\frac{y^2 - 5y}{y} \cdot \frac{4y}{y^2 - 25}$

4. $\frac{x^2 - x - 6}{x + 2} \cdot \frac{x + 5}{x - 3}$

5. $\frac{x^2 - 1}{x} \cdot \frac{x - 2}{x^2 - 3x + 2}$

6. $\frac{a^2 + 12a + 36}{a - 6} \cdot \frac{a^2 - 12a + 36}{a + 6}$

7. $\frac{9b + 9b^2}{9 - 9b^2} \cdot \frac{1 - b}{1 + b}$

8. $\frac{3m^2 + 2m}{m^3} \cdot \frac{m^3 - m}{m + 1}$

9. $\frac{7}{x} \cdot \frac{x^2 - 4}{x + 7} \cdot \frac{x^2 + 7x}{7x - 14}$

10. $(2m + 2)\left(\frac{m - 1}{5m}\right) \cdot \frac{19m^2}{m^2 - 1}$

11. $\frac{ax - 6x}{x^2} \cdot \frac{a^2 + 12a + 36}{a^2 - 36}$

12. $\frac{5x + 4}{2x} \cdot \frac{7x^2}{4x + 5}$

Problem Set 14-3b
(continued)

13. $\frac{a^2 + 5a + 6}{3a} \cdot \frac{5b}{a^2 + 9a + 20}$

14. $\frac{7y + 2}{5y - 2(2y) - y} \cdot \frac{15y - 7}{18y^2}$

Solve each of the following problems.

15. Separate 80 into two parts such that half of the larger part, increased by five times the smaller part, results in the sum, 76.

16. The sum of the digits of a two digit number is 12. The tens place digit is three times the units place digit. Find the number.

14-4. Adding Quotients of Polynomials.

The expression

$$\frac{7}{36a^2b} + \frac{5}{24b^3}$$

is the indicated sum of two "polynomial quotients." In this section, we shall work with the simplification of such expressions.

Just as with multiplication, the simplification is based on the way in which real numbers are added:

For any real numbers a, b, and c, $b \neq 0$,

$$\frac{a}{b} + \frac{c}{b} = \frac{a + c}{b} .$$

The following numerical example is similar to ones considered in Chapter 11. The steps involved in the simplification were carefully explained at that time.

Example 1. Simplify $\frac{5}{12} + \frac{1}{10}$.

$$\frac{5}{12} + \frac{1}{10} = \frac{5}{2 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 5}$$

The least common denominator is $2 \cdot 2 \cdot 3 \cdot 5$.

$$= \frac{5}{2 \cdot 2 \cdot 3}\left(\frac{5}{5}\right) + \frac{1}{2 \cdot 5}\left(\frac{2 \cdot 3}{2 \cdot 3}\right)$$

$$= \frac{25}{60} + \frac{6}{60}$$

$$= \frac{31}{60} .$$

Below are two examples involving variables. Once necessary restrictions have been placed on any variables involved, the expressions in these examples may be simplified in exactly the same way as in the numerical example above, since they represent real numbers.

Example 2. Simplify $\frac{3x}{6a} + \frac{5}{2a^2}$. $(a \neq 0)$

$$\frac{3x}{6a} + \frac{5}{2a^2} = \frac{3x}{2 \cdot 3 \cdot a} + \frac{5}{2 \cdot a \cdot a}$$

The least common denominator is $2 \cdot 3 \cdot a \cdot a$.

$$= \frac{3x}{2 \cdot 3 \cdot a}\left(\frac{a}{a}\right) + \frac{5}{2 \cdot a \cdot a}\left(\frac{3}{3}\right)$$

$$= \frac{3ax}{6a^2} + \frac{15}{6a^2}$$

$$= \frac{3ax + 15}{6a^2} .$$

The result is a quotient of polynomials.

Example 3. Simplify $\frac{7}{36a^2b} + \frac{5}{24b^3}$. $a \neq 0, b \neq 0$

$$\frac{7}{36a^2b} + \frac{5}{24b^3} = \frac{7}{2^2 \cdot 3^2 \cdot a^2 \cdot b} + \frac{5}{2^3 \cdot 3 \cdot b^3}$$

The least common denominator is $2^3 \cdot 3^2 \cdot a^2 \cdot b^3$.

$$= \frac{7}{2^2 \cdot 3^2 \cdot a^2 \cdot b} \cdot \frac{2b^2}{2b^2} + \frac{5}{2^3 \cdot 3 \cdot b^3} \cdot \frac{3a^2}{3a^2}$$

$$= \frac{14b^2}{72a^2b^3} + \frac{15a^2}{72a^2b^3}$$

$$= \frac{14b^2 + 15a^2}{72a^2b^3}$$

The result is a single indicated quotient of polynomials.

Check Your Reading

In questions 1 and 2, complete the statement, based on the reading and the examples of this section.

1. For any real numbers a, b, and c, with $b \neq 0$, $\frac{a}{b} + \frac{c}{b} =$ ___.
2. If $a \neq 0$, $\frac{3ax}{6a^2} + \frac{15}{6a^2} =$ _____.
3. If $2 \cdot 2 \cdot 3$ is the denominator of one fraction, and $2 \cdot 5$ is the denominator of a second fraction, what is the least common denominator?
4. If $2 \cdot 3 \cdot a$ is the denominator of one fraction, and $2 \cdot a \cdot a$ is the denominator of a second fraction, what is the least common denominator?
5. If $2^2 \cdot 3^2 \cdot a^2 \cdot b$ is the denominator of one fraction, and $2^3 \cdot 3 \cdot b^3$ is the denominator of a second fraction, what is the least common denominator?
6. If two quotients of polynomials are added, is the result also a quotient of polynomials?

Oral Exercises 14-4a

State the least common denominator of each of the following expressions: (It is assumed that the domains of the variables are correctly restricted.)

1. $\frac{1}{2}$, $\frac{1}{6a^2}$, $\frac{1}{3a^3}$
2. $\frac{1}{x + 2}$, $\frac{1}{x + 3}$
3. $\frac{1}{ab}$, $\frac{1}{a^2}$, $\frac{c}{b^2}$
4. $\frac{3}{c^2}$, $\frac{4}{ac}$, $\frac{5}{a^2c}$
5. $\frac{1}{x - 2}$, $\frac{1}{(x - 2)^2}$
6. $\frac{1}{3m^2}$, $\frac{1}{2mn}$, $\frac{1}{16m^3}$
7. $\frac{1}{4a}$, $\frac{1}{12a^2}$, $\frac{1}{24a}$
8. $\frac{1}{15ab}$, $\frac{1}{9a^2}$, $\frac{1}{5b^2}$
9. $\frac{1}{(m + 3)^2}$, $\frac{1}{2(m + 3)}$, $\frac{1}{8}$
10. $\frac{7}{36a^2b}$, $\frac{5}{24b^3}$
11. $\frac{5}{6a}$, $\frac{7}{30a^2}$, $\frac{11}{20b}$
12. $\frac{1}{2(a - b)}$, $\frac{1}{6(a - b)^2}$, $\frac{1}{15(a - b)}$

Problem Set 14-4a

Simplify each of the following expressions. Indicate the restrictions on the domains of the variables.

1. $\frac{2}{b} + \frac{3}{7b}$
2. $\frac{5}{2a} + \frac{7}{6a}$
3. $\frac{1}{3c} + \frac{1}{12c^2}$
4. $\frac{5}{12x} + \frac{1}{10x^2}$
5. $\frac{5a}{6y} + \frac{-2}{15y^2}$
6. $\frac{1}{3m^2} + \frac{1}{2mn}$
7. $\frac{3}{x - 1} + \frac{5}{3(x - 1)}$
8. $\frac{1}{2(x + 3)} + \frac{-x}{(x + 3)^2}$

Problem Set 14-4a
(continued)

9. $\frac{4}{5(a - b)} + \frac{a}{(a - b)(a + b)}$

10. $\frac{2a}{(a - b)^2} + \frac{-3}{(a - b)}$

11. $\frac{3}{x^2} + \frac{-2}{5x} + \frac{1}{25x}$

12. $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$

13. $\frac{4}{x(x - 1)} + \frac{3}{x - 1} + \frac{-1}{x}$

14. $\frac{5x}{(x - 3)(x + 3)} + \frac{7}{x + 3}$

15. $\frac{3}{m - 1} + \frac{2}{m - 2}$

16. $\frac{x}{x + 5} + \frac{-x}{x - 3}$

17. $\frac{x}{x + y} + \frac{-y}{x - y}$

18. $\frac{2}{a - b} + \frac{-3}{b - a}$

19. $\frac{y - 5}{2y} + \frac{y + 5}{y^2} + \frac{-1}{y^2}$

20. $\frac{5}{x - 1} + \frac{3}{x + 1} + 1$

Although the polynomials themselves may be quite complicated, the process of simplifying an indicated sum of polynomial quotients remains the same. The process may be outlined as follows:

1. Restrict the domain of the variables so that each quotient represents a real number.
2. Determine the least common denominator by expressing each denominator as a product of prime factors; then use the multiplication property of one to express each quotient in terms of this common denominator.
3. Apply the sentence: For all real numbers a, b, c, $b \neq 0$, $\frac{a}{b} + \frac{c}{b} = \frac{a + c}{b}$.
4. If possible use the multiplication property of one to simplify the result.

Example 1. Simplify $\dfrac{7}{y^2 + y - 12} + \dfrac{2}{y^2 - 8y + 15}$.

$$\frac{7}{y^2 + y - 12} + \frac{2}{y^2 - 8y + 15} = \frac{7}{(y + 4)(y - 3)} + \frac{2}{(y - 5)(y - 3)}$$

$y \neq -4,\ y \neq 3,\ y \neq 5$

$$= \frac{7}{(y + 4)(y - 3)}\left(\frac{y - 5}{y - 5}\right) + \frac{2}{(y - 5)(y - 3)}\left(\frac{y + 4}{y + 4}\right)$$

The least common denominator is $(y + 4)(y - 3)(y - 5)$.

$$= \frac{7(y - 5)}{(y + 4)(y - 3)(y - 5)} + \frac{2(y + 4)}{(y + 4)(y - 3)(y - 5)}$$

$$= \frac{7(y - 5) + 2(y + 4)}{(y + 4)(y - 3)(y - 5)}$$

$$= \frac{9y - 27}{(y + 4)(y - 3)(y - 5)}$$

This result may be simplified by the multiplication property of 1.

$$= \frac{9(y - 3)}{(y + 4)(y - 3)(y - 5)}$$

$(y - 3)$ is a factor of both numerator and denominator.

$$= \frac{y - 3}{y - 3} \cdot \frac{9}{(y + 4)(y - 5)}$$

"$\dfrac{y - 3}{y - 3}$" is a name for 1, if $y \neq 3$.

$$= \frac{9}{(y + 4)(y - 5)}$$

Now the numerator and denominator have no common factor.

Example 2. Simplify $\frac{1}{3x - 9} - \frac{2x}{5x - 15}$.

This expression indicates the subtraction of one polynomial quotient from another. However, if $x \neq 3$, both quotients represent real numbers; and the definition of subtraction may be used to restate the expression in terms of addition, as follows:

$$\frac{1}{3x - 9} - \frac{2x}{5x - 15} = \frac{1}{3x - 9} + \left(- \frac{2x}{5x - 15}\right)$$

$$= \frac{1}{3x - 9} + \frac{-2x}{5x - 15} .$$

If $x \neq 3$,

$$\frac{1}{3x - 9} + \frac{-2x}{5x - 15} = \frac{1}{3(x - 3)} + \frac{-2x}{5(x - 3)}$$

The least common denominator is $(3)(5)(x - 3)$.

$$= \frac{1}{3(x - 3)}\left(\frac{5}{5}\right) + \frac{-2x}{5(x - 3)}\left(\frac{3}{3}\right)$$

$$= \frac{5}{15(x - 3)} + \frac{-6x}{15(x - 3)}$$

$$= \frac{5 + (-6x)}{15(x - 3)}$$

$$= \frac{5 - 6x}{15(x - 3)} .$$

This result may also be written as "$\frac{5 - 6x}{15x - 45}$".

Problem Set 14-4b

Simplify each of the following expressions. Indicate the domains of the variables.

1. $\frac{1}{3x - 3} + \frac{5}{x}$
2. $\frac{13}{x} - \frac{2x}{x^2 + 2x}$

Problem Set 14-4b
(continued)

3. $\frac{1}{a^2} + \frac{a + 1}{a^2 - 4a}$

4. $\frac{3}{c^2} - \frac{c + 1}{c^2 + 2c}$

5. $\frac{4}{b} - \frac{b - 2}{b^2 - 5b}$

6. $\frac{3}{(b - 1)^2} + \frac{5}{7b - 7}$

7. $\frac{6}{y^2 + 3y} - \frac{y - 2}{(y + 3)(y - 3)}$

8. $\frac{3 - x^2}{x^2 - 1} + \frac{x + 2}{x + 1}$

9. $\frac{16}{y^2 + 2y - 8} + \frac{8}{y + 4}$

10. $\frac{3}{x^2 + 2x} - \frac{5}{3x + 6}$

11. $\frac{4}{m^2 - 7m + 12} + \frac{5}{m^2 + m - 20}$

12. $\frac{7}{a - b} + \frac{6}{a^2 - 2ab + b^2}$

13. $\frac{5}{x^2 - 1} - \frac{3}{x^2 + 2x + 1}$

14. $\frac{2x}{x^2 + 7x - 8} + \frac{3}{x^2 - 6x + 5}$

15. $\frac{v}{v^2 + v - 12} - \frac{1}{v^2 - 8v + 15}$

16. $\frac{3y}{y^2 - 2y - 8} + \frac{y}{y^2 - 4}$

17. $\frac{3x}{2x^2 - x - 1} - \frac{2}{x + 1}$

18. $\frac{5x}{3x^2 - 2x - 8} + \frac{3}{2x - 4}$

19. $\frac{a}{a^2 - 1} + \frac{5}{a - 1} + \frac{1}{a + 1}$

20. $\frac{z}{z^2 - 25} - \frac{2}{3z + 15} + \frac{3}{2z - 10}$

14-5. Rational Expressions and Rational Numbers.

Although the numbers 3 and 2 are integers, the quotient of these numbers, $\frac{3}{2}$, is not an integer; it is, however, a rational number. In fact, a rational number is any real number that can be named as the quotient of two integers.

At the beginning of this chapter, it was suggested that polynomials over the integers behave something like the integers themselves. If, instead of starting with two integers, we start with two polynomials, say "$2x + 1$" and "$x^2 - 3$," the indicated quotient of these polynomials may be written as

$$\frac{x^2 - 3}{2x + 1} .$$

This indicated quotient is not a polynomial. However, just as $\frac{3}{2}$ is called a rational number because it is the quotient of two integers, so is

$$\frac{x^2 - 3}{2x + 1}$$

called a rational expression because it is the quotient of two polynomials over the integers.

Thus, the indicated quotients of polynomials with which we worked in sections 14-3 and 14-4 may also be called rational expressions. Listed below are three more illustrations of rational expressions:

$$\frac{3}{5x^3 + 7} , \qquad \frac{2x^3 - x^2 + 4}{3x - 5} , \qquad 3x^2 - 7x - 6.$$

The first two are obviously quotients of polynomials. The third may not appear to be so. However, remember that an integer, say for example 5, is also a rational number since it can be expressed as the quotient $\frac{5}{1}$. In the same way, a polynomial over the integers, such as $3x^2 - 7x - 6$ is also a rational expression since it may be written

$$\frac{3x^2 - 7x - 6}{1} .$$

The following pairs of statements are very similar. One pair concerns integers and rational numbers. The other pair deals with polynomials and rational expressions.

> Every integer is also a rational number.
> However, not all rational numbers are integers.
>
> Every polynomial over the integers is also a rational expression.
> However, not all rational expressions are polynomials.

Although every rational number can be named as the quotient of two integers, it is possible to use names for rational numbers that are not quotients of integers. For example, starting with $\frac{3}{2}$, we might add 7, like this:

$$\frac{3}{2} + 7$$

The expression above is not the quotient of two integers. However, it does name a rational number; it can be simplified to "$\frac{17}{2}$".

In the same way, we might begin with the polynomial quotient $\frac{x^2 - 3}{2x + 1}$ and add $(x + 2)$, like this:

$$\frac{x^2 - 3}{2x + 1} + x + 2.$$

The resulting expression is not the indicated quotient of two polynomials. But it is considered to be a rational expression. It can be simplified to the indicated quotient of two polynomials.

It is possible to write quite complicated names for both rational numbers and rational expressions, as the illustrations below suggest.

$$\frac{-5}{2 - \frac{1 + 2}{6}}$$

This numeral names a rational number. It can be simplified to read as the quotient of two integers.

$$\frac{-x^2 + x - 3}{2x + 1 - \frac{2}{x - 3}}$$

This expression is a rational expression. It can be simplified to read as the quotient of two polynomials over the integers.

The above illustration of a rational expression suggests the following definition:

> A rational expression is one which indicates at most the operations of addition, subtraction, multiplication, division, and taking opposites.

Compare this with the definition of a polynomial in Chapter 13. They differ only in that the operation of division is not permitted in a polynomial but is permitted in a rational expression.

The phrase "at most," occurring in the definition, means that no operations other than the five mentioned in the definition can be indicated in a rational expression. It does not mean, of course, that a rational expression must indicate all five of the operations.

Example 1. Is $\frac{\sqrt{2x + 3}}{x - 5}$ a rational expression?
It is not, since the operation of "taking a square root" is indicated, and this operation is not one of the five permitted in a rational expression.

Notice that the definition above does not say that every rational expression is the quotient of two polynomials over the integers, since this is not true. However, it is true that every rational expression can be simplified to a form indicating the quotient of two polynomials over the integers.

Example 2. Simplify $\frac{2x^2 + \frac{1}{3}x}{5 - 2x}$. $x \neq \frac{5}{2}$

This expression is already the indicated quotient of two polynomials; however, they are not both polynomials over the integers. The multiplication property of one may be used to obtain a simplification.

$$\frac{2x^2 + \frac{1}{3}x}{5 - 2x} \cdot \frac{3}{3} = \frac{(2x^2 + \frac{1}{3}x)(3)}{(5 - 2x)(3)}$$

$$= \frac{6x^2 + x}{15 - 6x}.$$

This expression is the indicated quotient of two polynomials over the integers.

Example 3. Simplify $(1 + \frac{1}{x - 1})(x - 1)$.

This is a rational expression. The only operations indicated are addition, subtraction, multiplication, and division; all of these are included in the definition of a rational expression.

However, the expression is not in simplest form, since it is not the indicated quotient of two polynomials over the integers.

If $x \neq 1$, then $\frac{1}{x - 1}$, as well as "1" and "$x - 1$", represents a real number. The distributive property may be applied, as follows:

$$\begin{aligned}(1 + \frac{1}{x - 1})(x - 1) &= (1)(x - 1) + (\frac{1}{x - 1})(x - 1) \\ &= (x - 1) + 1 \\ &= x.\end{aligned}$$

Check Your Reading

1. What name is given to numbers which can be expressed as quotients of integers?
2. What name is given to expressions which can be written as quotients of polynomials over the integers?
3. Which of the following statements is true:
   Every polynomial is a rational expression;
   Every rational expression is a polynomial ?
4. In the definition of a rational expression, five permissible operations are stated. What are they?
5. Is the expression $\dfrac{-x^2 + x - 3}{2x + 1 - \frac{2}{x - 3}}$ a quotient of polynomials over the integers? Is it a rational expression?

Oral Exercises 14-5

1. Give an example of a rational expression which is a polynomial. Explain why it is a polynomial.
2. Give an example of an expression which is not a rational expression. Explain why it is not.
3. Give an example of a rational expression in three variables.
4. For each of the following expressions state whether it is

   (i) a polynomial
   (ii) a rational expression
   (iii) neither of these

   (a) $a^2 + 2ab + b^2$
   (b) $\frac{a + b}{a + b}$
   (c) $2a + 5b$
   (d) $x^2 + 2x + 1 = 0$
   (e) $\frac{5x^2 + 2}{3x - 2x - x}$
   (f) $\frac{2x^2 - 7}{x + 1}$
   (g) $\sqrt{x^2 + y^2}$
   (h) $\left(\sqrt{x^2}\right)^2$
   (i) $|a + b|$
   (j) $5x + 7 = 2x + 4$
   (k) $\frac{x + y}{x - y}$
   (l) $(x - 5)(x + 2) + (2x - 3)$
   (m) $\sqrt{a} - \sqrt{a}$
   (n) $\frac{a^2 + 2a + 5}{a + 2}$
   (o) $\frac{(x + 5)^3}{x + 5}$

Problem Set 14-5

1. State whether each of the following is

   (i) a polynomial
   (ii) a rational expression
   (iii) a sentence

   and in how many variables it is written.

   (a) $a^2 + 2ab + b^2$
   (b) $(x + 5)^2(x - 4)^3$
   (c) $4(2m + 3)^3(m - 7)$
   (d) $\frac{a^2 + 2ab + b^2}{a + b}$

Problem Set 14-5
(continued)

(e) $\frac{3x(x - 5)}{4x^2}$

(f) 0

(g) $5(2x + 4)^2$

(h) $|a + b|$

(i) $x^2 + 2x - 1 = 5$

(j) $\frac{3}{4}$

(k) $(x + y + z)^5$

(l) $7x(y + m)$

(m) $7 + 5$

(n) $\frac{3x + 2y - k}{(x + y) - (x - y)}$

(o) $\left(\sqrt{a + b + c}\right)^2$

(p) $\sqrt{x} = 3$

(q) $\frac{3 + \frac{1}{x}}{x - 2}$

2. Write an expression in one variable which is
   (a) a polynomial
   (b) a rational expression but not a polynomial
   (c) a non-rational expression.

3. Simplify. State the domains of the variables.

(a) $\frac{\frac{1}{2}m + \frac{1}{3}}{\frac{1}{3}m}$

(b) $\frac{3 + \frac{1}{a}}{\frac{1}{a}}$

(c) $(1 - \frac{1}{b})b$

(d) $\frac{3}{x^2} + \frac{5}{x} + 1$

(e) $\frac{1 + \frac{1}{y}}{\frac{3}{y^2}}$

(f) $\frac{\frac{1}{x - 1}}{\frac{2}{x + 1}}$

(g) $(3 + \frac{1}{x + 2})(x + 2)$

(h) $\frac{3y^2 + 6}{\frac{3}{y}}$

(i) $\frac{\frac{3}{uv^2}}{u}$

(j) $\frac{3 + \frac{1}{y}}{3 - \frac{1}{y}}$

14-6. Dividing Polynomials.

$\frac{2x^2 + x - 5}{x - 3}$ is a rational expression in simplest form; it is the indicated quotient of two polynomials over the integers, and the numerator and denominator do not have a common factor. Nevertheless, there are times when it is desirable to change this expression to another form. Let us first look at a similar situation in arithmetic, where both polynomials are simply integers---for example, $\frac{168}{11}$.

"$\frac{168}{11}$" is a name for a rational number. Another name for this number is "$15\frac{3}{11}$." In arithmetic, you may have spoken of this as "changing from an improper fraction to a mixed number." The "mixed number" name may be obtained by long division, as follows:

$$\begin{array}{r} 15 \\ 11\overline{)\,168} \\ \underline{11} \\ 58 \\ \underline{55} \\ 3. \end{array}$$

The division process above is well known from arithmetic. However, there are certain points about this process that should be understood before a corresponding process for division of any two polynomials is discussed. For this reason, the following form of the process is given.

$11\overline{)\,168}$	⑩	Take away 10 "11's" from 168.
$\underline{-110} \leftarrow$		(10)(11) = 110.
58	⑤	Next, take away 5 more "11's."
$\underline{-55} \leftarrow$		(5)(11) = 55.
3		3 is less than 11. Therefore, no more "11's" are taken away. 3 is the remainder.

The explanation above shows that division may be thought of as repeated subtraction. "11's" were subtracted from 168 until a number smaller than 11 was obtained. First, 10 "11's" were subtracted, then 5 "11's." The number 3 remained.

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Therefore, we may write

$$168 - (10)(11) - (5)(11) = 3$$

$$168 = 10(11) + 5(11) + 3$$

$$168 = (10 + 5)(11) + 3$$

$$\underset{\underline{\text{dividend}}}{168} = \underset{\underline{\text{quotient}}}{(15)}(\underset{\underline{\text{divisor}}}{11}) + \underset{\underline{\text{remainder}}}{3}$$

The underlined words are names commonly used in division.

The sentence on the right below may be obtained from the sentence on the left by multiplying both sides by 11. And the sentence on the left may be obtained from the sentence on the right by multiplying both sides by $\frac{1}{11}$. The sentences are equivalent.

$$\frac{168}{11} = 15 + \frac{3}{11} \qquad\qquad 168 = (15)(11) + 3$$

In fact, the sentence on the right represents a familiar method of "checking" division: multiply the quotient by the divisor, and add the remainder; the result should be the dividend.

Following is another example, in which division is approached from the point of view of subtraction. Again, the reason for this lies in preparation for division of polynomials.

$24\overline{)\ 821}$	㉚	$(30)(24) = 720$. 720 is subtracted from 821.
$\underline{720}$		
101	④	$(4)(24) = 96$. 96 is subtracted.
$\underline{96}$		
5		5 is less than 24. Therefore, the remainder is 5.

The same division process is exhibited below in a way familiar from arithmetic. The process is really the same; notice how the "30" and the "4" show up.

$$\begin{array}{r} 34 \\ 24\,\overline{)\,821} \\ \underline{720} \\ 101 \\ \underline{96} \\ 5 \end{array}$$

The following statements are equivalent:

$$\frac{821}{24} = 34 + \frac{5}{24}; \qquad \underset{\text{dividend}}{821} = \underset{\text{quotient}}{(34)}\underset{\text{divisor}}{(24)} + \underset{\text{remainder}}{5}$$

In general, if n and d are positive integers and $n \leq d$, then $\frac{n}{d}$ may be expressed in "mixed number" form by the long division process, indicated roughly by the following diagram:

$$\begin{array}{r} q \\ d\,\overline{)\,n} \\ \\ \overline{r}. \end{array}$$

For any n and d satisfying the above conditions, a number q can be found such that $0 \leq r < d$, where r and q are integers

The following true statements can then be made:

$$\frac{n}{d} = q + \frac{r}{d}, \qquad n = qd + r.$$

Check Your Reading

1. What process is used to change "$\frac{168}{11}$" to the "mixed number" name "$15\frac{3}{11}$"?
2. In this section, it is pointed out that division of integers may be thought of as a repeated application of what operation?
3. In the statement "168 = (15)(11) + 3," identify the dividend, the divisor, the quotient, and the remainder.
4. The remainder is always less than what number?
5. Give a statement that is equivalent to "168 = (15)(11) + 3."
6. Give a statement that is equivalent to "$\frac{n}{d} = q + \frac{r}{d}$."

Oral Exercises 14-6a

1. Tell how $\frac{15}{4}$ can be written in several ways as the sum of a positive integer and a positive rational number. Which of these is in the form $\frac{n}{d} = q + \frac{r}{d}$, $0 \leq r < d$?

2. If in a division process the quotient is 2, the remainder is 2 and the divisor is 5, what is the dividend?

3. Write $\frac{16}{19}$ as the sum of an integer and a rational number.

4. Find the values of the variables for which the following sentences are true.

$n = q \cdot d + r$	$\frac{n}{d} = q + \frac{r}{d}$
(a) $15 = 3 \cdot d + 0$	(f) $\frac{12}{5} = q + \frac{2}{d}$
(b) $27 = q \cdot 10 + 7$	(g) $\frac{37}{d} = 5 + \frac{2}{d}$
(c) $49 = 9 \cdot 5 + r$	(h) $\frac{81}{11} = 7 + \frac{r}{d}$
(d) $n = 8 \cdot 6 + 2$	(i) $\frac{112}{5} = 22 + \frac{r}{d}$
(e) $93 = 2 \cdot d + 5$	(j) $\frac{n}{5} = 11 + \frac{2}{d}$

Problem Set 14-6a

Perform each of the following indicated divisions; then write the results in each of the forms $n = qd + r$ and $\frac{n}{d} = q + \frac{r}{d}$.

1. $\frac{229}{17}$
2. $\frac{486}{23}$
3. $\frac{192}{131}$
4. $\frac{768}{47}$
5. $\frac{18}{9}$
6. $\frac{41}{10}$
7. $\frac{22}{5}$
8. $\frac{45}{15}$

9. A man walked a distance in miles that was 4 times the number of hours he walked. Write an expression for his rate of speed.

Problem Set 14-6a
(continued)

10. A stack of quarters was divided into three equal piles with a pile of 2 remaining. The total value of the quarters was \$29.75. How many were in each pile? Translate this into an open sentence and find its truth set.

$\frac{168}{11}$ is the indicated quotient of two integers.

It may be changed to "mixed number" form by the long division process.

$\frac{2x^2 + x - 5}{x - 3}$ is the indicated quotient of two polynomials over the integers.

The similarity between the behavior of integers and of polynomials over the integers suggests that this rational expression may be changed to a form corresponding to "mixed number" form, by a long division process.

Let us follow the long division process for integers, and apply it to the polynomials "x - 3," the divisor, and "$2x^2 + x - 5$," the dividend.

$$x - 3\,\overline{\big)\,2x^2 + x - 5} \quad \textcircled{2x}$$

$$\underline{2x^2 - 6x}$$

Subtract $(2x)(x - 3)$. This is suggested by the fact that $(2x)(x) = 2x^2$. $(2x)(x - 3) = 2x^2 - 6x$.

$$7x - 5 \quad \textcircled{7}$$

$$\underline{7x - 21}$$

$$16$$

Subtract $(7)(x - 3)$. This is suggested by the fact that $(7)(x) = 7x$. $(7)(x - 3) = 7x - 21$. 16 is the remainder.

Do you see that the subtraction approach was used with division of polynomials, just as it was with positive integers in the previous section? First $(2x)(x - 3)$ was subtracted, then $(7)(x - 3)$ was subtracted; the number 16 remained. Thus,

$$2x^2 + x - 5 - (2x)(x - 3) - (7)(x - 3) = 16;$$

$$2x^2 + x - 5 = (2x)(x - 3) + (7)(x - 3) + 16;$$

$$\underbrace{2x^2 + x - 5}_{\text{dividend}} = \underbrace{(2x + 7)}_{\text{quotient}}\underbrace{(x - 3)}_{\text{divisor}} + \underbrace{16}_{\text{remainder}}.$$

Provided that $x \neq 3$, the following two sentences are equivalent:

$$2x^2 + x - 5 = (2x + 7)(x - 3) + 16;$$

$$\underbrace{\frac{2x^2 + x - 5}{x - 3}}_{\text{This is a rational expression}} = \underbrace{2x + 7 + \frac{16}{x - 3}}_{\text{This is another form of the expression, corresponding to the "mixed number" form of arithmetic.}}.$$

In dividing positive integers, we stopped whenever the remainder was less than the divisor. You may wonder why we stopped with 16 in the example above, since it makes no sense to say that 16 is "less" than the divisor $x - 3$. When dividing polynomials other than two integers, the remainder must be of degree less than the degree of the divisor. In the example above,

The divisor $x - 3$ is of degree one.
The remainder 16 is of degree zero.

Below is another example.

Change the rational expression $\frac{8x^2 + 4x - 2}{2x - 7}$ to another form by the long division process.

$$\begin{array}{r l}
2x - 7\,\overline{)\,8x^2 + 4x + 2} & \enclose{circle}{4x} \\
\underline{8x^2 - 28x} & \\
32x + 2 & \enclose{circle}{16} \\
\underline{32x - 112} & \\
114 &
\end{array}$$

Subtract $(4x)(2x - 7)$, suggested by the fact that $(4x)(2x) = 8x^2$. $(4x)(2x - 7) = 8x^2 - 28x$.

Subtract $(16)(2x - 7)$, suggested by the fact that $(16)(2x) = 32x$. $(16)(2x - 7) = 32x - 112$.

The degree of "114" is less than the degree of the divisor "2x - 7." Therefore, 114 is the remainder.

In this example, $(4x)(2x - 7)$ was subtracted from $8x^2 + 4x + 2$. Then $(16)(2x - 7)$ was subtracted. Altogether, $(4x)(2x - 7) + (16)(2x - 7)$ was subtracted. That is, using the distributive property, $(4x + 16)(2x - 7)$ was subtracted, with remainder 114. Thus,

$$8x^2 + 4x + 2 = (4x + 16)(2x - 7) + 114;$$

$$\frac{8x^2 + 4x + 2}{2x - 7} = 4x + 16 + \frac{114}{2x - 7}.$$

Check Your Reading

1. When dividing positive integers, it was stated that the remainder is less than the divisor. What corresponding statement is made in this section concerning division of polynomials other than integers?
2. What is the degree of "x - 3"?
3. What is the degree of "16"?

Check Your Reading
(continued)

4. $2x^2$ is the product of x and what other factor?

5. $8x^2$ is the product of $2x$ and what other factor?

6. If $x - 3$ is the divisor, identify the dividend, the quotient, and the remainder in the sentence "$2x^2 + x - 5 = (2x + 7)(x - 3) + 16$."

7. If $x - 3$ is the divisor, give a sentence which is equivalent to $2x^2 + x - 5 = (2x + 7)(x - 3) + 16$.

Oral Exercises 14-6b

1. Give the degree of the following polynomials.

   (a) $x^2 + 2x + 1$ (d) 15
   (b) $3x^3 - 5x + 1$ (e) $2x^2$
   (c) $3x$ (f) $-3x + 5$

2. Give the first term of the quotient for each of the following:

   (a) $\dfrac{8x^2 + 5x + 3}{x + 1}$ (d) $3a - 1\,\overline{)\,9a^2 + 5}$
   (b) $x + 7\,\overline{)\,5x^2 + 4}$ (e) $\dfrac{-7a + 3}{a - 5}$
   (c) $\dfrac{4a^2 - 6a + 5}{2a + 2}$ (f) $4m + 3\,\overline{)\,12m^3 + 5m - 3}$

3. In each of the following subtract the bottom polynomial from the top polynomial.

   (a) $\begin{array}{l} 5x^3 - 3x^2 \\ \underline{5x^3 + 3x^2} \end{array}$ (c) $\begin{array}{l} -3x^2 + 2x + 1 \\ \underline{-3x^2 - 2x - 1} \end{array}$

   (b) $\begin{array}{l} -2y^3 + 3y^2 + 4y + 2 \\ \underline{-2y^3 + 4y^2 \qquad\qquad} \end{array}$ (d) $\begin{array}{l} 11x^2 + 7x \\ \underline{11x^2 - 7x + 2} \end{array}$

Oral Exercises 14-6b
(continued)

(e) $11x^2 + 7$
$\underline{11x^2 - 7x + 2}$

(f) $7x^4 - 3x^2$
$\underline{7x^4 + 2x^3}$

(g) $3y^3 - 3y^2$
$\underline{2y^3 + y}$

(h) $5x^3 + 3x - 2x^2 + 7$
$\underline{5x^3 - 2x^2}$

(i) $3x^3 - 2x^2 + x + 1$
$\underline{3x^3 - 2x^2 + x + 1}$

(j) $\frac{1}{2}x^3 - \frac{2}{3}x^2 + x - \frac{3}{8}$
$\underline{\frac{1}{2}x^3 + \frac{1}{2}x^2}$

Problem Set 14-6b

1. The following sentences are given in the form $\frac{n}{d} = q + \frac{r}{d}$. For each write an equivalent sentence in the form $n = q \cdot d + r$. Identify the dividend, the divisor, the quotient, and the remainder.

   (a) $\frac{5x^2 - 7x - 10}{x - 2} = 5x + 3 + \frac{-4}{x - 2}$

   (b) $\frac{x^3 + 2x^2 - x + 10}{x + 3} = x^2 - x + 2 + \frac{4}{x + 3}$

   (c) $\frac{x^3 + 2x^2 + 2x - 1}{x + 1} = x^2 + x + 1 + \frac{-2}{x + 1}$

2 The following sentences are given in the form $n = q \cdot d + r$. For each write an equivalent sentence in the form $\frac{n}{d} = q + \frac{r}{d}$. Identify the dividend, the divisor, the quotient, and the remainder.

   (a) $x^3 - 2x^2 + 2x + 1 = (x^2 - x + 1)(x - 1) + 2$

   (b) $x^3 + 4x^2 + 3x - 5 = (x^2 + 2x - 1)(x + 2) + (-3)$

   (c) $3x^2 - 10x - 3 = (3x + 2)(x - 4) + 5$

Problem Set 14-6b
(continued)

3. State the monomial needed in each of the following indicated multiplications so that the indicated subtraction results in a lower degree polynomial. Write the resulting polynomial.

   (a) $(x^3 + 2x^2) - (?)(x + 1)$

   (b) $(4x^4 - 3x^3) - (?)(x + 3)$

   (c) $(-5x^2 + 3x + 4) - (?)(x - 2)$

   (d) $(6x^3 + 2x^2) - (?)(3x + 1)$

   (e) $(12x^5 - 2x^2) - (?)(4x^2 + 2x + 1)$

   (f) $(x^2 + 2x + 1) - (?)(x + 1)$

   (g) $(3x^2 + 5x - 7) - (?)(x - 3)$

   (h) $(4x^3 - 7x^2 + 2x + 4) - (?)(x - 1)$

   (i) $(3x^3 - 4x + 7) - (?)(x^2 - 3x + 2)$

   (j) $(2x^5 - 3x^4 + 2x^3 - x^2 - x + 2) - (?)(x^3 + 2x^2 - 3x + 1)$

The long division process for two polynomials may be shown in a way exactly like that used in arithmetic to show the long division process for two integers. Examples 1 and 2 illustrate this.

Example 1. Divide $8x^2 + 4x + 2$ by $2x - 7$.

This problem is the same as the last example in the previous section. Compare the work below with the work done at that time.

$$\begin{array}{rl}
 & \,4x + 16 \\
2x - 7 & \overline{\big)\, 8x^2 + 4x + 2} \\
 & \,\underline{8x^2 - 28x} \\
 & \,32x + 2 \\
 & \,\underline{32x - 112} \\
 & 114
\end{array}$$

Example 2. Change $\frac{x^3 - 38x - 10}{x - 5}$ to another form by long division.

$$
\begin{array}{r|rrrr}
 & x^2 & +5x & -13 & \\
\hline
x - 5 & x^3 & & -38x & -10 \\
 & \underline{x^3} & \underline{-5x^2} & & \\
 & & 5x^2 & -38x & -10 \\
 & & \underline{5x^2} & \underline{-25x} & \\
 & & & -13x & -10 \\
 & & & \underline{-13x} & \underline{+65} \\
 & & & & -75
\end{array}
$$

Note that there is no "x^2" term in the dividend. Hence, as a matter of convenience, a space is left.

Therefore, provided $x \neq 5$,

$$\frac{x^3 - 38x - 10}{x - 5} = x^2 + 5x - 13 + \frac{-75}{x - 5}.$$

The long division process may be used to answer questions about factoring, as illustrated in Examples 3 and 4.

Example 3. Is 63 a factor of 567?

$$
\begin{array}{r|r}
 & 9 \\
\hline
63 & 567 \\
 & \underline{567} \\
 & 0
\end{array}
$$

The remainder is zero.

$$
\begin{aligned}
\text{Therefore, } 567 &= (63)(9) + 0 \\
&= (63)(9)
\end{aligned}
$$

63 is a factor of 567.

Example 4. Is $x - 1$ a factor of $x^3 - 3x^2 + 3x - 1$?

$$
\begin{array}{r|rrrr}
 & x^2 & -2x & +1 & \\
\hline
x - 1 & x^3 & -3x^2 & +3x & -1 \\
 & \underline{x^3} & \underline{-x^2} & & \\
 & & -2x^2 & +3x & -1 \\
 & & \underline{-2x^2} & \underline{+2x} & \\
 & & & x & -1 \\
 & & & \underline{x} & \underline{-1} \\
 & & & & 0
\end{array}
$$

The remainder is zero.

$$\text{Therefore, } x^3 - 3x^2 + 3x - 1 = (x^2 - 2x + 1)(x - 1) + 0$$
$$= (x^2 - 2x + 1)(x - 1).$$

$x - 1$ is a factor of $x^3 - 3x^2 + 3x - 1$.

Oral Exercises 14-6c

1. How does the division process $\frac{n}{d} = q + \frac{r}{d}$ tell you when d is a factor of n?

2. In which of the following cases is d a factor of n? Why?

   (a) $\frac{x^2 - 7x + 12}{x - 3} = x - 4$

   (b) $\frac{x^2 + 6x + 10}{x + 3} = (x + 3) + \frac{1}{x + 3}$

   (c) $\frac{2x - 2}{x - 1} = 2$

   (d) $\frac{14x^2 + 12x - 2}{14x - 2} = x + 1$

   (e) $\frac{10x^2 - 7x - 10}{2x - 3} = 5x + 4 + \frac{2}{2x - 3}$

   (f) $\frac{x^4 - 4x^2 - 4x - 1}{x^2 - 2x - 1} = x^2 + 2x + 1$

Problem Set 14-6c

Perform each of the following indicated divisions until the remainder is of lower degree than the divisor. In each case check your result by multiplication and addition, guided by the form $n = qd + r$.

1. $\frac{x^2 + 5x + 6}{x + 3}$

2. $\frac{4x^2 - 4x + 1}{2x - 1}$

3. $\frac{2x^2 - 4x + 3}{x - 2}$

4. $\frac{5x^3 + 4x^2 - 3x + 7}{x + 1}$

5. $\frac{x^3 - 3x^2 + 7x - 1}{x - 3}$

6. $\frac{x^4 - 9x^2 - 1}{x + 3}$

Problem Set 14-6c
(continued)

7. $\frac{5x^3 - 11x + 7}{x + 2}$

8. $\frac{x^3 + 1}{x + 1}$

9. $\frac{x^4 - 1}{x - 1}$

10. $\frac{2x^3 - 2x^2 + 5}{x - 6}$

11. $\frac{2x^5 + x^3 - 5x^2 + 2}{x - 1}$

12. $\frac{x^5 + 1}{x + 1}$

13. $\frac{9x^2 + 12x + 4}{3x + 2}$

*14. $\frac{6x^3 - x^2 - 5x + 4}{3x - 2}$

15. What polynomial multiplied by $x + 2$ results in the polynomial $3x^2 + 13x + 14$?

16. What polynomial multiplied by $x - 5$ results in the polynomial $2x^3 - 9x^2 - 9x + 20$?

17. What polynomial multiplied by $x + 3$ results in the polynomial $3x^3 + 9x^2 - x - 3$?

18. Find the missing factor in each of the following:
    (a) $(x + 5)(?) = x^2 + 7x + 10$
    (b) $(x - 7)(?) = 3x^2 - 16x - 35$
    (c) $(x - 3)(?) = 5x^2 - 16x + 3$
    (d) $(2x + 4)(?) = 4x^2 + 16x + 16$
    (e) $(x + 3)(?) = x^2 - 9$

Summary

1. While the sum, difference, and product of two polynomials are always polynomials, the quotient of two polynomials is not always a polynomial.

2. In working with quotients of polynomials it is necessary to restrict the domains of the variables to exclude values for which the denominator is zero.

3. Simplification of an indicated product of two quotients of polynomials can be accomplished by applying the sentence,

   "For all real numbers a, b, c, and d, $b \neq 0$, $d \neq 0$,

   $$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}\text{,"}$$

   and then using the multiplication property of one to simplify the result.

4. To simplify an indicated sum of two quotients of polynomials we

   determine the least common denominator,
   express each quotient in terms of this common denominator,
   apply the sentence:

   "For all real numbers a, b, and c, $b \neq 0$,

   $$\frac{a}{b} + \frac{c}{b} = \frac{a + c}{b}\text{."}$$

   use the multiplication property of one to simplify the results.

5. A rational expression is one which indicates at most the operations of addition, subtraction, multiplication, division, and taking of opposites.

6. A rational expression can be expressed as a quotient of polynomials.

7. If p and d are polynomials in one variable such that the degree of p is not less than the degree of d, then there are polynomials q and r satisfying the condition

   $$p = qd + r,$$

   where the degree of r is less than the degree of d.

Review Problem Set

1. Which of the following are rational expressions? polynomials? Which are polynomials in one variable? polynomials over the integers? over the rational numbers? over the real numbers?

   (a) $(s^2 - t)(3st + 1) + 5(s + t)$

Review Problem Set
(continued)

(b) $7x^2 + 2x - 5$

(c) $ax^2 + bx + c$

(d) $2u + v$

(e) $2(u + \frac{1}{2}v)$

(f) $u + \frac{1}{2}v$

(g) $\frac{1}{2}(4u + 2v)$

(h) $(\frac{1}{x})^2 - 2(\frac{1}{x}) + 1$

(i) $\sqrt{x}$

(j) $\sqrt[3]{x}$

(k) $(|x| + 1)(|x| - 1)$

(l) $\frac{a - b}{a + b}$

(m) $\frac{a^2 - b^2}{a + b}$

(n) $\frac{r - 5}{s + 2} \cdot \frac{r + 5}{s - 2}$

(o) $\frac{r - 5}{s + 2} \cdot \frac{s + 2}{r - 5}$

(p) $\frac{1}{3 - z} + \frac{z}{2 + z}$

(q) $s^2 - 2$

(r) $(s + \sqrt{2})(s + \sqrt{2})$

2. Simplify each of the following expressions.

(a) $\sqrt{18}$

(b) $3\sqrt{2} + 2\sqrt{2}$

(c) $\sqrt{\frac{3}{4}}$

(d) $\sqrt{2} \cdot \sqrt{6}$

(e) $\sqrt{a^2}$

(f) $2\sqrt{18} + 3\sqrt{12}$

(g) $\sqrt{\frac{1}{2}} - 6\cdot\sqrt{\frac{1}{3}}$

(h) $\sqrt{3} \cdot \sqrt{a^4}$

(i) $\sqrt{(x + y)^2}$

(j) $\sqrt{\frac{x^2}{2}}$

3. Simplify each of the following expressions.

(a) $3(\sqrt{2} + 3)$

(b) $\sqrt{2}\left(\sqrt{6} + \sqrt{\frac{1}{2}}\right)$

(c) $2\sqrt{3}(2 - \sqrt{6})$

(d) $(\sqrt{3} + \sqrt{2})^2$

(e) $(\sqrt{2} + \sqrt{5})(\sqrt{2} - \sqrt{5})$

Review Problem Set
(continued)

4. Factor each of the following polynomials into prime factors over the integers.

(a) $3a + 6ab$

(b) $x + y + 3x + 3y$

(c) $3mx - 9x + 12xy - 3x$

(d) $x^2 - 11x + 30$

(e) $x^2 - 22x - 48$

(f) $x^2 - 22x + 48$

(g) $(a + b)(a - b) - 4(a - b)$

(h) $x^2 - y^2 - 4x - 4y$

(i) $3a^3b^5 - 6a^2b^3 + 12a^4b^4$

*(j) $6a^2 - 19a + 10$

*(k) $6a^2 + 11a - 10$

(l) $64x^2 - 100a^2$

(m) $32a^2 - 162$

*(n) $m^2 + 2mn + n^2 - a^2$

5. Simplify each of the following expressions.

(a) $\frac{1}{21} + \frac{5}{33}$

(b) $\frac{3}{a} - \frac{2}{b}$

(c) $\frac{5a}{x} + \frac{2b}{3x}$

(d) $\frac{3}{35a^2} + \frac{13}{25ab}$

(e) $\frac{2x}{x - 2} - \frac{3}{4x}$

(f) $\frac{2}{a - 2} - \frac{5a}{2a^2 - 4a}$

(g) $\frac{x}{x^2 - 9} - \frac{3x}{x^2 + 2x - 3}$

(h) $\frac{2m + 1}{m + 2} + \frac{7}{m - 2} - \frac{3m}{m^2 - 4}$

6. Perform each of the following indicated divisions and check the results.

(a) $\frac{3x^2 - 19x - 40}{x - 8}$

(b) $\frac{2x^2 + 3x + 5}{x - 1}$

(c) $\frac{x^3 - 4x^2 + x + 6}{x - 3}$

(d) $\frac{x^5 - 1}{x - 1}$

(e) $\frac{10x^2 + 9x - 9}{5x - 3}$

Review Problem Set
(continued)

7. Find the truth set of each of the following sentences.

(a) $2x + 5x - 4 = x - \frac{1}{2}$

(b) $x - 7 < 3x + 2$

(c) $\frac{2x}{3} - \frac{x}{5} = \frac{1}{3}$

(d) $\frac{1}{x} + \frac{3}{2x} = 6$

(e) $(x - 7)(4x + 5) = 0$

(f) $x^2 - 13x + 40 = 0$

(g) $\frac{3}{5x} - \frac{3}{4x} = \frac{1}{10}$

(h) $\frac{4}{y-5} = \frac{5}{4}$

(i) $\frac{x + 2}{x + 1} - \frac{x - 1}{x + 2} = 0$

(j) $4x^2 - 243 = x^2$

(k) $\frac{14}{|x - 3|} = 7$

(l) $\frac{5}{n - 3} - \frac{20}{n^2 - 9} = -1$

*(m) $3|x|^2 - 2|x| = 0$

Hint: factor

*(n) $|x|^2 + |x| = 12$

*(o) $|x - 5|^2 \geq 9$

Hint: Consider separately the cases $(x - 5) \geq 0$ and $(x - 5) < 0$.

8. Show whether or not $(x - 3)$ is a factor of the polynomial $2x^2 + x - 20$.

9. Which of the following numbers are rational?

(a) $\sqrt[3]{\pi^3}$

(b) $\sqrt{.04}$

(c) $\sqrt[3]{-1}$

(d) $\sqrt{\frac{81}{100}}$

(e) $\sqrt{11}$

Review Problem Set
(continued)

10. Find the average of $\frac{x + 3}{x}$ and $\frac{x - 3}{x}$, $x \neq 0$.

Solve each of the following problems.

11. The square of a number is 91 more than 6 times the number. What is the number?

12. The sum of the reciprocals of two successive integers is $\frac{15}{56}$. What are the integers?

13. One leg of a right triangle is 2 feet more than twice the smaller leg. The hypotenuse is 13 feet. Find the length of each leg.

14. A jet travels 10 times as fast as a passenger train. In one hour the jet will travel 120 miles farther than the passenger train will go in 8 hours. What is the rate of each?

15. A candy store made a 40 lb. mixture part of which had cream centers and sold at $1.00 per pound while the rest had nut centers and sold at $1.40 per pound. The final mixture is to sell for $1.10 per pound. How many pounds of each kind of candy should there be?

16. Two trains 160 miles apart travel towards each other. One is traveling $\frac{2}{3}$ as fast as the other. How fast is each going if they meet in 3 hours and 12 minutes?

Chapter 15

TRUTH SETS OF OPEN SENTENCES

15-1. Equivalent Open Sentences.

As seen earlier, equivalent sentences are open sentences with the same truth set. The truth set of an open sentence may be found by forming a chain of equivalent sentences, a process illustrated below in finding the truth set of "$3x + 8 = 20$."

$$3x + 8 = 20$$

$$3x + 8 + (-8) = 20 + (-8)$$

-8 is added to both sides, resulting in a sentence equivalent to the original one.

$$3x = 12$$

$$\left(\frac{1}{3}\right)3x = \left(\frac{1}{3}\right)12$$

Each side is multiplied by $\frac{1}{3}$, resulting in a sentence equivalent to the one above.

$$x = 4$$

The truth set of the last sentence, "$x = 4$," is clearly $\{4\}$. However, since we formed a chain of equivalent sentences, $\{4\}$ is also the truth set of the original sentence in the chain, "$3x + 8 = 20$."

Do you see how the chain of equivalent sentences was formed in the above example? Two ideas, studied earlier and reviewed below, were applied.

$a = b$ is equivalent to $a + c = b + c$, for any real number c.	The same number may be added to both sides without changing the truth set.
$a = b$ is equivalent to $ca = cb$, for any non-zero real number c.	Both sides may be multiplied by the same non-zero number without changing the truth set.

Even though we know that equivalent sentences have the same truth set, it is still wise to "check" truth numbers in the original sentence, as a guard against arithmetic mistakes.

Thus, in the above example, the number 4 may be checked in the sentence "$3x + 8 = 20$," as follows:

$$3(4) + 8 = 20.$$

This sentence is true (both sides are names for the number 20), showing that 4 is a truth number of "$3x + 8 = 20$."

As a second example in which equivalent sentences occur, consider the open sentence

$$4y + 7 = y + 28.$$

$4y + 7 + (-7) = y + 28 + (-7)$ -7 is added to both sides.

$4y = y + 21$

$4y + (-y) = y + 21 + (-y)$ $-y$ is added to both sides.

$3y = 21$

$(\frac{1}{3})3y = (\frac{1}{3})21$ Both sides are multiplied by $\frac{1}{3}$.

$y = 7$

In the chain of equivalent sentences above, notice that in one step -7 was added to both sides, and in another step $-y$ was added to both sides. Actually, these steps may be combined into a single step by adding $-7-y$ to both sides, as follows:

$4y + 7 = y + 28$

$4y + 7 + (-7-y) = y + 28 + (-7-y)$ $-7-y$ is added to both sides.

$3y = 21$

$(\frac{1}{3})3y = (\frac{1}{3})21$ Both sides are multiplied by $\frac{1}{3}$.

$y = 7$

In either case, we arrive at the sentence "$y = 7$," whose truth set is easily seen to be $\{7\}$. This is also the truth set of the original sentence, "$4y + 7 = y + 28$."

When we first solved open sentences by forming chains of equivalent sentences, the truth set was verified by "reversing" the chain, that is, by reversing the steps in the chain. If this were done in the last example above, we would start with

"$y = 7$," then go "up the ladder" to obtain "$4y + 7 = y + 28$." However, such a demonstration is not necessary if we can be sure in advance that each step taken is reversible. The question is, then, what steps are of this type? In other words, what kinds of steps can we reverse, or "undo," by another step? As one example, the step of "adding 5" may be reversed by the step of "adding -5." As a matter of fact, we have already learned that:

> Addition of any real number is reversible. Such a step can be reversed by adding the opposite.
>
> Multiplication by any non-zero real number is reversible. Such a step can be reversed by multiplying by the reciprocal. The number zero has no reciprocal; that is the reason we must exclude multiplication by zero as a reversible step.

The solution of "$4y + 7 = y + 28$" above illustrated both of these kinds of steps. The number $(-7-y)$ was added to both sides. This is a reversible step, since it may be reversed by adding the opposite of $(-7-y)$, that is, by adding $7 + y$. Also, both sides were multiplied by $\frac{1}{3}$. This is a reversible step; it can be reversed by multiplying by the reciprocal of $\frac{1}{3}$, that is, by multiplying by 3.

Check Your Reading

1. What is meant by the phrase "equivalent sentences"?
2. State two ways in which a sentence equivalent to a given sentence may be obtained.
3. What step reverses the step of adding 5?
4. Is the addition of any real number a reversible step?
5. What step reverses addition of $(-y-7)$?
6. What step reverses multiplication by $\frac{1}{3}$?
7. Is multiplication by any real number a reversible step?

Oral Exercises 15-1a

1. Describe in each case the step which reverses the given step.

   Examples: adding 7, reverse step: adding -7;
   multiplying by 4, reverse step: multiplying by $\frac{1}{4}$.

   (a) adding 6
   (b) adding -10
   (c) multiplying by 5
   (d) adding $\frac{2}{3}$
   (e) adding $(y + 7)$
   (f) multiplying by $\frac{3}{4}$
   (g) adding $(-x-6)$
   (h) multiplying by $\frac{1}{10}$
   (i) multiplying by -6
   (j) adding w

2. Give the truth sets of the following sentences.

   (a) $2x = 8$
   (b) $x + 5 = 6$
   (c) $3w = 6$
   (d) $2y + 1 = 7$
   (e) $5x = 20$
   (f) $3y = 1$
   (g) $2w + 5 = 5$
   (h) $4x = 3$

Problem Set 15-1a

1. Two sentences are equivalent if they have the same truth set. Show whether or not the following pairs of sentences are equivalent by comparing truth sets.

   (a) $3x = 6;\quad 4x = 8$
   (b) $5t = -10;\quad t + 6 = 44$
   (c) $\frac{1}{2}t = 3;\quad 2 = \frac{1}{3}t$
   (d) $12 = 4x;\quad 3x = 12$

2. Two sentences are equivalent if one can be obtained from the other using reversible steps. For the following pairs of sentences show that the second can be obtained from the first and the first can be obtained from the second.

Problem Set 15-1a
(continued)

Example: $5t + 7 = 10; \quad 5t = 3$

Solution: $5t = 3$ is obtained from $5t + 7 = 10$ by adding (-7) to each side.
$5t + 7 = 10$ is obtained from $5t = 3$ by adding 7 to each side.

(a) $3x - 8 = 12; \quad 3x = 20$

(b) $4y = 12; \quad y = 3$

(c) $9t + 7 = 2t; \quad 7t + 7 = 0$

(d) $7x - 2 = 3x + 5; \quad 4x - 2 = 5$

(e) $3 + 5 = 3t + 5; \quad 2 = 2t$

(f) $\frac{3}{4}h - 1 = h; \quad 3h - 4 = 4h$

(g) $\frac{k}{4} + \frac{1}{2} = \frac{k}{2} - \frac{1}{4}; \quad k + 2 = 2k - 1$

(h) $3x^2 = 27; \quad x^2 = 9$

(i) $14 + n = 25 + 3n; \quad 1 = 12 + 2n$

3. For each of the following pairs of sentences, determine whether or not the sentences are equivalent. You can show this by beginning with either sentence and carrying out operations that yield equivalent sentences, until you arrive at the other sentence of the pair. If you think they are not equivalent, try to show it by finding a number that is in the truth set of one, but not in the truth set of the other.

(a) $2x = 12; \quad x = 6$
(b) $12 + 3n = 5n; \quad 12 = 2n$
(c) $5y - 4 = 3y + 8; \quad y = 6$
(d) $7s - 5s = 12; \quad s = 6$
(e) $3x + 9 - 2x = 7x - 12; \quad \frac{7}{3} = x$
(f) $2x^2 + 4 = 10; \quad x^2 = 4$

Problem Set 15-1a
(continued)

4. For each pair of sentences below, decide whether or not the sentences are equivalent.
   - (a) $4 - 2x = 10; \quad x = -3$
   - (b) $12x + 5 = 10 - 3x; \quad x = \frac{1}{3}$
   - (c) $w^2 - 4 = 0; \quad (w - 2)(w + 2) = 0$
   - (d) $3x - 2 - 4x + 6 = 0; \quad -x + 4 = 0$
   - (e) $0 = t - 3; \quad 4(t - 3) = 0$
   - (f) $3y - 3 = 0; \quad y = 1$
   - (g) $0 = x^2 - 2x; \quad 0 = x(2 - x)$
   - (h) $2(h + 2) + 2(h + 3) = 27; \quad 4h + 10 = 27$
5. Solve (that is, find the truth set of):
   - (a) $3x + 6 = 12$
   - (b) $8 = 5x - 2$
   - (c) $6 - y = 7$
   - (d) $32 = 11t + 21$
   - (e) $6 - s = s + 6$
   - (f) $s - 6 = 6 - s$
   - (g) $s - 6 = s + 6$
   - (h) $\frac{y}{3} = 5$
   - (i) $y + 2 = 3y - 6$
   - (j) $\frac{x}{2} + \frac{1}{2} = 5$
   - (k) $x^2 + 3x = x^2 - 9$
   - (l) $\frac{5}{8}x - 17 = 33$

In the sentences which we have been solving so far we have obtained equivalent sentences by addition and multiplication. We have been able to add expressions which contain a variable, such as $x + 7$ or $x - 3$, and get equivalent sentences. This is true because we know that such phrases represent real numbers for all values of the variable. We also know that addition to both sides is permissible for any real number.

However, in the case of multiplication, we have been multiplying only by expressions which do not contain a variable. We could be sure of obtaining an equivalent sentence as long as the number we were multiplying by was not zero.

We will now look at some examples which involve multiplication by an expression which *does* contain a variable. We will discover, in such cases, that we do not always obtain an equivalent sentence. For example, consider the following open sentence.

$$x(x - 3) = 2(x - 3)$$

If we experiment a bit, we will see that this sentence has the truth set $\{2, 3\}$. Now, suppose we had wanted to use the method we have been working with, namely that of obtaining equivalent sentences. It would have been natural to multiply both sides by the expression

$$\frac{1}{x - 3}$$

This would give us

$$x(x - 3) \frac{1}{x - 3} = 2(x - 3) \frac{1}{x - 3} ,$$

which becomes

$$x = 2.$$

But this last sentence has the truth set $\{2\}$. Is this last sentence equivalent to the first? It should be clear that the answer is no, since the truth set of the first sentence is $\{2, 3\}$.

The question is, "What happened?"

To answer this we must look carefully at the expression "$\frac{1}{x - 3}$". Remember that our rule states that if we multiply both sides of an open sentence by a non-zero *real number*, we will obtain an equivalent sentence. Now it is true that the expression $\frac{1}{x - 3}$ does represent a real number for most values of the variable x. For example, if x has the value 5, then the expression is a numeral for $\frac{1}{2}$. Do you see this?

On the other hand, suppose x has the value 3. In this case we see that the denominator has the value 0. What does this mean? It means that our expression does *not* represent a real number when $x = 3$.

In other words, the phrase $\frac{1}{x - 3}$ does *not* represent a real number for *all* values of the variable. This helps to explain why the second sentence was not equivalent to the first.

Let's consider another example. In this case we will not be attempting to solve a sentence. We will, however, learn some more about equivalence. We begin with the open sentence

$$w = 5.$$

We now multiply both sides by w and obtain the sentence

$$w^2 = 5w.$$

Question: "Is the second sentence equivalent to the first?" The best way to answer this is to look at the two truth sets. The truth set of the first is clearly $\{5\}$. What is the truth set of the second sentence? If we experiment, we will find that it has two elements. The set is $\{5, 0\}$, because

$5^2 = 5(5)$ and $0^2 = 5(0)$ are both true.

Our answer, then, must be that the two sentences are not equivalent.

Again we must look at the multiplier. It is the variable w. Surely this always represents a real number. But w can also have the value 0, and our rule states that we will always obtain an equivalent sentence if we multiply both sides by a non-zero real number.

The above examples have shown us that we do not always obtain equivalent sentences when we multiply both sides by an expression which contains a variable. In fact, in both of our problems we did not obtain an equivalent sentence. It is natural to ask if this always happens. The answer, as we will soon see, is no. We often obtain equivalent sentences even when the multiplier does contain a variable. The important thing to remember is that we must in all of these cases be especially careful to check the truth sets, since we cannot always be sure that the final sentence is equivalent to the original one.

Check Your Reading

1. Does $\frac{1}{x - 3}$ represent a real number for all values of x?
2. Are the open sentences "$x(x - 3) = 2(x - 3)$" and "$x = 2$" equivalent? Why or why not?
3. Are the open sentences "$w^2 = 5w$" and "$w = 5$" equivalent? Why or why not?
4. If both sides of an open sentence are multiplied by an expression involving a variable, is the resulting sentence necessarily equivalent to the original one? Give two examples from this section that support your answer.

Oral Exercises 15-1b

For each of the pairs of sentences below, explain why they are equivalent or why they are not equivalent.

1. $x = 3; \quad 3x = 9$
2. $x = 3; \quad x^2 = 3x$
3. $7x = 7; \quad x = 1$
4. $\frac{x}{7} = \frac{1}{7}; \quad x = 1$
5. $7x^2 = 7x; \quad 7x = 7$
6. $t = 1; \quad t^2 = t$
7. $m^2 = m; \quad m = 1$
8. $5(y - 1) = y(y - 1); \quad 5 = y$
9. $6 = m; \quad 6(m - 1) = m(m - 1)$
10. $x(x + 1) = 0; \quad x + 1 = 0(\frac{1}{x})$

Problem Set 15-1b

1. For each of the following phrases, find values of the variable for which the phrase is (i) zero, (ii) not a real number.

   (a) $x - 3$

   (b) $y + 4$

   (c) $\frac{1}{t - 2}$

   (d) $\frac{1}{h}$

   (e) x

   (f) $(t - 2)(t - 3)$

   (g) $\frac{1}{t(t + 1)}$

   (h) $\frac{1}{(x - 1)(x + 1)}$

Problem Set 15-1b
(continued)

2. For each of the following phrases, decide whether the phrase is
   - (i) zero for some value of the variable,
   - (ii) not a real number for some value or values of the variable,
   - (iii) a non-zero real number for every value of the variable.

(a) $y + 5$

(b) x^2

(c) $x^2 + 1$

(d) $\frac{x}{7}$

(e) $\frac{1}{y - 6}$

(f) $\frac{1}{y^2 + 1}$

(g) $(x - 1)(x - 2)(x - 3)$

3. You can write your own practice exercises by choosing a sentence which has an obvious truth set and then building up a more complex equivalent sentence.
For example:

$x = 2$

$x + 7 = 9$ adding 7 to each side

$3x + 7 = 9 + 2x$ adding $2x$ to each side

For each of the following sentences, write a more complex sentence by performing the given operations. Then decide if the operations performed guarantee that the new sentence is equivalent to the given sentence and state your reason. (If in doubt, check by finding truth sets.)

(a) $x = 2$; add $x + 3$ to each side.
(b) $3 = x$; multiply each side by 4.
(c) $y = 4$; add $y - 2$ to each side.
(d) $x = 1$; multiply each side by x.
(e) $w = 0$; multiply each side by $(w - 3)$.
(f) $x = 10$; add 15 to each side and multiply each side by $\frac{x}{5}$.

Problem Set 15-1b
(continued)

4. Which of the following operations on a sentence might not yield an equivalent sentence? State a reason for your decision.
   (a) Multiply each side by $(x - 1)$.
   (b) Add $3x + 7$ to each side.
   (c) Multiply each side by x.
   (d) Multiply each side by $\frac{1}{x - 1}$.
   (e) Multiply each side by $\frac{1}{x}$.
   (f) Add $\frac{1}{x}$ to each side.

5. Solve.
   (a) $5x + 3 = 2x + 12$
   (b) $x(x - 2) = 0$
   (c) $0 = n^2 - 3n$
   (d) $y^2 - 4y = 0$
   (e) $\frac{x}{5} - \frac{1}{3} = \frac{x}{3} - 1$
   (f) $\frac{p}{7} + 1 = 3$
   (g) $2t + 7 = 3 + 2t$
   (h) $7(y - 3) - 3 = 5(y + 3) + (y^2 + 5)0$
   (i) $x + 3 = 3x - 6$

We will now consider some sentences whose sides, or members, contain rational expressions such as

$$\frac{1}{x}, \quad \frac{x + 3}{x + 2}, \quad \frac{3}{x + 7}, \quad \frac{1}{x - 3}.$$

Before we begin solving such sentences, however, it is important that we review some ideas about the domain of the variable. You will recall that the domain is the set of numbers from which the value of the variable may be chosen. It was pointed out that the domain sometimes depends on the type of problem being solved. For instance, if a problem involves finding the number of people, we rule out fractions and negative numbers. The domain then becomes the set of whole numbers.

We have agreed in general that unless something special is said, the domain will be the real numbers. We further state that if we are given an open sentence, then the domain will be the set of all real numbers for which the sentence has a meaning. For example, the sentence

$$3x + 5 = 26$$

has a meaning for all real numbers. Therefore, in a sentence of this type, we will assume the domain to be the set of all real numbers.

However, suppose we look at the following sentence.

$$\frac{3}{x - 5} = 12.$$

Do you see that the sentence has no meaning if x is 5, since this would give us 0 for the denominator? We will therefore assume that the domain of the variable of this particular open sentence is the set of all real numbers except the number 5. For the sentence

$$\frac{1}{x} = 5$$

we would say that the domain is the set of all real numbers except 0. Do you see why?

By now it should be clear that the domain of the variable of the sentence

$$\frac{x}{x - 4} + \frac{3}{x + 3} = \frac{5}{x}$$

is the set of all real numbers except 4, -3, and 0.

We will now return to the problem of finding truth sets of open sentences whose members contain rational expressions. Consider the sentence

$$\frac{12}{x - 3} = 6.$$

We first see that the domain of x cannot include the number 3. We now wish to multiply both sides by the expression $(x - 3)$. The question is, "Will we then get an equivalent sentence?".

To answer this, we note that in this particular problem the domain does not include 3. Therefore $(x - 3)$ represents a non-zero real number for all values of x in the domain. We can now multiply and obtain

$$\frac{12}{x-3}(x - 3) = 6(x - 3)$$

which becomes $$12 = 6x - 18.$$

Adding 18 to both sides we get

$$30 = 6x.$$

Multiplication by $\frac{1}{6}$ finally gives us

$$5 = x. \qquad (\text{Domain: } x \neq 3)$$

We can now say that the sentence

$$5 = x \qquad (\text{Domain: } x \neq 3)$$

is equivalent to the sentence we started with. The truth set of "$5 = x$" is $\{5\}$. Since 5 is in our domain, we know then that $\{5\}$ is the truth set we want. It is important that the final sentence be accompanied by a statement about the domain of the variable. This is the domain determined by the original sentence.

Let us look at a second example.

$$\frac{8}{n} = 2.$$

This sentence tells us that our domain must exclude the real number 0. With 0 excluded from the domain, we can multiply by n and obtain an equivalent sentence. This gives us

$$8 = 2n.$$

We now multiply both sides by $\frac{1}{2}$ and get the sentence

$$4 = n. \qquad (\text{Domain: } x \neq 0)$$

Its truth set is $\{4\}$. This sentence is equivalent to the original sentence. We know then that $\{4\}$ is the truth set of the original sentence.

We see from the above examples that it is very important to examine the sentence we start with before we begin working the problem. In this way we can see what the domain must be.

It will then be possible to obtain a sentence which is equivalent to the first. From this we can determine the truth set of the original sentence.

A final example should clear up this point. Consider the open sentence

$$\frac{x}{x - 2} = \frac{2}{x - 2} .$$

The sentence tells us that the domain for this problem must exclude the number 2. We now proceed to multiply by $(x - 2)$ with this in mind. From this we get

$$\frac{x}{x - 2}(x - 2) = \frac{2}{x - 2}(x - 2)$$

which becomes $x = 2$. (Domain: $x \neq 2$)

This sentence is equivalent to the one we started with. The sentence "$x = 2$" has a truth set $\{2\}$ only in domains which include the number 2. Therefore, in the domain of this particular problem, the truth set of

$$x = 2 \qquad \text{(Domain: } x \neq 2\text{)}$$

is empty. From this we can say that the truth set of our original sentence is also empty. We see now why it is important that a statement about domain should accompany the final sentence.

Check Your Reading

1. What is the domain of the variable in each of the following open sentences:

   $3x + 5 = 26$, $\frac{3}{x - 5} = 12$, $\frac{1}{x} = 5$, $\frac{x}{x - 4} + \frac{3}{x + 3} = \frac{5}{x}$.

2. What is the domain of x in the sentence "$\frac{12}{x - 3} = 6$"?

3. Does "$x - 3$" represent a non-zero real number for all values of x in the domain of the sentence $\frac{12}{x - 3} = 6$?

4. Is the sentence "$\frac{8}{n} = 2$" equivalent to the sentence "$8 = 2n$"?

Check Your Reading
(continued)

5. Are "$\frac{x}{x - 2} = \frac{2}{x - 2}$" and "$x = 2$" equivalent?

6. Are "$\frac{x}{x - 2} = \frac{2}{x - 2}$" and "$x = 2$" equivalent if 2 is excluded from the domain of the variable in the second problem?

Oral Exercises 15-1c

1. State the domain of each of the following sentences.

(a) $\frac{3}{x - 2} = 5$

(b) $\frac{5}{m} = m - 1$

(c) $8(t - 2) = 5$

(d) $\frac{1}{q(q + 1)} - \frac{1}{q + 2} = 5$

(e) $\frac{y}{5} - \frac{y - 1}{3} = \frac{1}{7}$

(f) $\frac{x}{x - 1} - \frac{2}{x - 2} = 3$

2. For each of the following explain how the second sentence may be obtained from the first and why equivalence is preserved by this process.

(a) $\frac{1}{x} = 5$; $1 = 5x$

(b) $3(x - 1) = 6$; $x - 1 = 2$

(c) $\frac{3}{x - 2} = 6$; $3 = 6(x - 2)$

(d) $3x + 5 = 2x$; $x = -5$

Problem Set 15-1c

1. For each of the following rational expressions, state the values of the variable for which the expression has no meaning (that is, does not represent a real number).

(a) $\frac{3}{x - 2}$

(b) $\frac{1}{x + 1}$

(c) $\frac{x}{x}$

(d) $\frac{7x}{x - 3}$

(e) $\frac{14}{x^2 - 1}$

(f) $\frac{1}{x^2 + 1}$

Problem Set 15-1c
(continued)

2. For each of the following state the domain of the variable such that the rational expression represents a real number.

   (a) $x - 2$

   (b) $\frac{10}{t}$

   (c) $\frac{5}{t(t - 5)}$

   (d) $\frac{y - 3}{y - 3}$

   (e) $\frac{k - 2}{k(k - 1)}$

   (f) $\frac{6}{(m - 6)(m + 5)}$

3. **Explain how the second sentence can be obtained from the first and why equivalence is preserved by this process.**

   Example: $\frac{1}{x - 2} = 3; \qquad 1 = 3(x - 2)$

   Multiply each side by $(x - 2)$, which is a non-zero real number for every x, since $x \neq 2$.

   (a) $\frac{5 - m}{5 + m} = 7m + 2; \qquad 5 - m = (7m + 2)(5 + m)$

   (b) $\frac{1}{(y - 1)(y - 2)} = 3; \qquad 1 = 3(y - 1)(y - 2)$

   (c) $\frac{1}{|x|} = 2; \qquad 1 = 2|x|$

   (d) $\frac{x - 2}{(x - 2)(x - 3)} = 1; \qquad \frac{1}{x - 3} = 1$

4. Solve:

   (a) $\frac{1}{y} = 3$

   (b) $\frac{3}{z} = 1$

   (c) $\frac{1}{m + 2} = 3$

   (d) $6 = \frac{m}{m - 2}$

   (e) $\frac{1}{x} + 1 = 3$

   (f) $\frac{5}{x - 2} = \frac{1}{2}$

5. Solve:

   (a) $\frac{x}{x - 3} = \frac{4}{x - 3}$

   (b) $\frac{x}{x - 3} = \frac{3}{x - 3}$

   (c) $\frac{4}{|x|} = 1$

   (d) $\frac{t}{t - 2} = \frac{5}{3}$

   (e) $\frac{t - 1}{t + 1} = \frac{7}{9}$

   (f) $\frac{3m + 5}{m + 1} = \frac{4 + 2m}{m + 1}$

A sentence may have a variable in the denominator and still have as a domain the set of all real numbers. The following example is of this type. Consider the open sentence

$$\frac{3x^2 + x}{x^2 + 1} = 3.$$

Here we see that the sentence has a meaning when the value of x is any real number. Do you see the reason for this? Is it clear that the denominator will never be equal to 0?

We may multiply both sides by the expression $(x^2 + 1)$ without having to exclude any real number from the domain. This gives us

$$\frac{3x^2 + x}{x^2 + 1}(x^2 + 1) = 3(x^2 + 1) ,$$

which becomes

$$3x^2 + x = 3x^2 + 3.$$

We may now add $(-3x^2)$ to both sides. This will give us

$$x = 3.$$

(If no statement about the domain is given, we always assume that it is the set of all real numbers for which the sentence has meaning.)

This sentence is equivalent to the first one. Why? We therefore know that the truth set of

$$\frac{3x^2 + x}{x^2 + 1} = 3 \quad \text{is} \quad \{3\}.$$

Sentences which contain rational expressions such as the ones we have been studying are often called fractional equations. In the previous chapter you have used the idea of a least common multiple, or least common denominator, in working with rational expressions. We use this in solving the following fractional equation:

$$\frac{1}{x} = \frac{1}{1 - x} .$$

Here we note that the domain is the set of all real numbers excluding both 0 and 1. We also see that the least common denominator is the expression $x(1 - x)$. If we keep in mind the special domain for this problem, we may obtain an equivalent sentence by multiplying both sides by $x(1 - x)$. This gives us

$$\frac{1}{x}\left(x(1 - x)\right) = \frac{1}{1 - x}\left(x(1 - x)\right)$$

which becomes

$$1 - x = x .$$

Adding x to both sides we get

$$1 = 2x.$$

Multiplication of both sides by $\frac{1}{2}$ gives us finally

$$\frac{1}{2} = x. \qquad \text{(Domain: } x \neq 1, \ x \neq 0)$$

We know that this sentence is equivalent to the one we started with. The truth set is therefore $\{\frac{1}{2}\}$.

Check Your Reading

1. What is the smallest value of the expression $x^2 + 1$?
2. Explain why "$\frac{3x^2 + x}{x^2 + 1} = 3$" is equivalent to "$3x^2 + x = 3(x^2 + 1)$."
3. In solving the sentence $\frac{1}{x} = \frac{1}{1 - x}$ each side of the equation is multiplied by the least common denominator of the fractions. Thus, each side is multiplied by what expression?
4. What values of x are excluded from the domain in the sentence $\frac{1}{x} = \frac{1}{1 - x}$? Is $x(1 - x)$ zero for any values of x from the domain?

Oral Exercises 15-1d

For each of the following sentences, state the domain of the variable. By what expression would you multiply both sides so that each member of the resulting equation would be a polynomial?

1. $\frac{1}{x} = 5$
2. $\frac{1}{t} = \frac{1}{2}$
3. $\frac{3}{m - 1} = \frac{2}{3}$
4. $\frac{1}{a} + \frac{1}{2a} = \frac{1}{2}$
5. $\frac{t}{t - 1} + \frac{1}{t} = 5$
6. $\frac{y - 1}{y} + \frac{2}{y + 1} = 1$
7. $\frac{1}{x} + \frac{1}{2} = \frac{1}{3x}$
8. $t + \frac{1}{t} = 3$
9. $m + \frac{2}{3m} = 5$
10. $\frac{x - 2}{x(x + 1)} = 7$

Problem Set 15-1d

In each of the following begin by first indicating all the real numbers which can not be in the domain.

1. Solve:
   (a) $\frac{2}{x} - \frac{3}{x} = 10$
   (b) $\frac{n}{2} = 10 + \frac{n}{3}$
   (c) $\frac{1}{3} - \frac{1}{x} = \frac{2}{x}$
   (d) $\frac{1}{y} = \frac{3}{y^2}$
   (e) $\frac{1}{3} + \frac{2}{|t|} = \frac{3}{|t|}$
   (f) $\frac{1}{m} = \frac{1}{2m} + 7$
2. Solve:
   (a) $\frac{3}{t} - \frac{2}{t} = \frac{1}{6}$
   (b) $\frac{2}{3y} + \frac{14}{3} = \frac{3}{y}$
   (c) $\frac{2}{7t} - \frac{5}{14} = \frac{1}{t}$
   (d) $\frac{2}{x - 3} + \frac{1}{x} = \frac{6}{x}$
   (e) $\frac{1}{t} = \frac{1}{t - 1}$
   (f) $\frac{x}{15} - \frac{x}{3} = \frac{4}{45}$
   (g) $\frac{13}{6w} = \frac{3}{2w} + \frac{2}{3w}$

Problem Set 15-1d
(continued)

3. Solve:

(a) $5x + 17 = 3x + 23$

(b) $\frac{x - 2}{x - 3} = \frac{x + 4}{x - 1}$

(c) $m + \frac{4m - 10}{m - 3} = \frac{m^2 - 9}{m - 3}$

(d) $\frac{2x^2 + x}{x^2 + 1} = 2$

(e) $0 = y(y - 3)$

(f) $\frac{3x - 6}{x} = 3 - \frac{4}{x - 1}$

(g) $\frac{5}{|x| + 4} = \frac{10}{26}$

4. Solve:

(a) $\frac{x - 1}{x + 1} + \frac{7}{3x} = 1$

(b) $\frac{t}{t + 2} = \frac{t + 3}{t - 1}$

(c) $\frac{m^2 + 6}{m^2 + 5} = 1 - \frac{2m}{m^2 + 5}$

(d) $\frac{x^2 + x}{x^2 + 3} = 1$

(e) $y + 1 = \frac{y^2 + y}{y + 1}$

(f) $3x + \frac{16x}{x - 5} = \frac{3x^2 + 5}{x - 5}$

(g) $\frac{3}{y^2 + 81} + 1 = \frac{3}{y^2 + 81}$

15-2. Equations Involving Factored Expressions.

In Chapter 13 we solved equations of the form

$$(x - 3)(x + 2) = 0.$$

Its truth set was found to be $\{3, -2\}$. This is the combined truth set of two sentences:

$$x - 3 = 0$$

$$x + 2 = 0$$

We can now think of this in another way. You remember that in Chapter 3 we studied two kinds of compound sentences. One of these had the connecting word "and". The other had the connecting word "or". An example of this second type would be

$$x - 3 = 0 \quad \text{or} \quad x + 2 = 0.$$

We learned that this type of compound sentence, with connecting word "or", is true if at least one clause is true; otherwise it is false. In other words the truth set of our compound sentence is

$$\{3, -2\},$$

since "$x - 3 = 0$" is true when $x = 3$; likewise "$x + 2 = 0$" is true when $x = -2$. From this we can see that the compound sentence

$$x - 3 = 0 \quad \text{or} \quad x + 2 = 0$$

is equivalent to our original sentence

$$(x - 3)(x + 2) = 0.$$

We can solve sentences which contain more than two factors. A sentence such as

$$(x - 3)(x + 2)(x - 2) = 0$$

is true if $(x - 3) = 0$

or if $(x + 2) = 0$

or if $(x - 2) = 0$.

Otherwise, it is false. Do you see that the truth set is

$$\{3, -2, 2\} ?$$

Check Your Reading

1. State a compound open sentence that is equivalent to "$(x - 3)(x + 2) = 0$."
2. Name the elements in the truth set of "$(x - 3)(x + 2)(x - 2) = 0$."

Oral Exercises 15-2a

1. Which of the sentences, $x + 3 = 0$, $x - 4 = 0$, and $(x + 3)(x - 4) = 0$, are true if $x = -3$? if $x = 4$?
2. How do you know that the sentence $(x + 3)(x - 4) = 0$ is false if x is 5?

Oral Exercises 15-2a
(continued)

3. What is the truth set of $x + 3 = 0$? of $x - 4 = 0$?

4. What is the truth set of $(x + 3)(x - 4) = 0$?

Problem Set 15-2a

1. Find the truth sets of:
   (a) $(x + 2)(x - 5) = 0$
   (b) $(t - 3)(t + 1) = 0$
   (c) $0 = (m - 6)(m - 6)$
   (d) $(k + 1)(k - 1) = 0$
   (e) $y(y + 3) = 0$
   (f) $(z - 1)^2 = 0$

2. Solve:
   (a) $(2x - 1)(x + 1) = 0$
   (b) $(x^2 + 1)(4x - 3) = 0$
   (c) $y^2(y - 1)^2 = 0$
   (d) $(2z + 3)(3z + 2) = 0$
   (e) $(2x - 1)(|x| - 1) = 0$
   (f) $(2y - 1)(|y - 1|) = 0$

3. Solve:
   (a) $(y - 1)(y - 2)(y - 3) = 0$
   (b) $x(x + 1)(x - 1) = 0$
   (c) $m(2m - 1)(3m + 2) = 0$
   (d) $(n - 1)(n + 1)n^2 = 0$
   (e) $(t^2 - 1)(2t - 3) = 0$
   (f) $(x^2 + 1)^2 = 0$

4. Find the truth sets of:
   (a) $x^2 - x - 2 = 0$
   (b) $0 = 121 - y^2$
   (c) $t^3 - 25t = 0$
   (d) $y^2 - y - 12 = 0$
   (e) $m^2 + 5m + 6 = 0$
   (f) $k^2 - 2 = 0$

5. Solve:
   (a) $x^2 + x = 2x^2$
   (b) $y^2 + 6y = 16$
   (c) $2m^2 - 5m - 3 = 0$
   (d) $x^2 + 2 = 0$
   (e) $3y^2 = 21y - 18$
   (f) $2m^2 + 7m = 15$

6. Solve: (Before solving indicate the domain of the variable.)
   (a) $x + \frac{1}{x} = \frac{10}{3}$
   (b) $y - \frac{2}{y} = 1$
   (c) $4 + \frac{7}{t - 3} = \frac{3(t - 1)}{t - 3}$
   (d) $\frac{1}{m} + \frac{2}{m - 3} = \frac{6}{5}$
   (e) $y + \frac{6}{y - 3} = -4$
   (f) $\frac{1}{z} - \frac{1}{z - 4} = 1$

Problem Set 15-2a
(continued)

7. For what integers is it true that the reciprocal of the integer is one-fourth the integer? Write and solve an open sentence to answer this question.

8. An integer is added to its reciprocal and the sum is $\frac{10}{3}$. For what integers is this sentence true? Write and solve an open sentence to answer this question.

9. A river boat can make a trip 120 miles downstream in the same time that it takes to make a trip 60 miles upstream. If the boat travels 15 miles per hour in still water, find the rate of the current.
   (a) Write an open sentence to answer this question.
   (b) What is the domain of the variable in the sentence for this problem?
   (c) Solve the sentence.
   (Hint: If s is the number of miles per hour in the rate of the current, then $15 + s$ is the number of miles per hour in the rate of the boat as it goes downstream, and $\frac{120}{15 + s}$ is the number of hours that it takes the boat to make the trip downstream. By a similar line of thinking, represent the number of hours for the trip upstream. Then you can write an open sentence that says that the time for the trip downstream is the same as the time for the trip upstream.)

10. The front wheel of a wagon has a circumference that is 3 feet less than that of the back wheel. If the front wheel makes as many turns in going 60 feet along the road as the back wheel does going 90 feet, find the circumference of each wheel.
    (a) Write an open sentence to answer the above problem.
    (b) What is the domain of the variable in your sentence for this problem?
    (c) Solve the sentence and give the answer to the problem.

In the early part of the chapter we discussed the danger of multiplying by an expression or adding an expression which for some value of the variable might not be a real number. However, in the case of problems of the type

$$x(x - 3) = 2(x - 3)$$

the domain is the set of all real numbers; in this example we noted the following danger. We were tempted to multiply by

$$\frac{1}{x - 3}$$

which would change the original sentence to

$$x = 2.$$

We saw that the truth set of the first sentence was $\{3, 2\}$. This means that the last sentence, whose truth set is $\{2\}$ is not equivalent to the first. The change in truth sets came about because our multiplier

$$\frac{1}{x - 3}$$

is not a real number for $x = 3$; but our domain does include the number 3.

The question then comes up, "What can we do to avoid changing the truth set in sentences of this type?". In other words, how can we keep the sentences equivalent?

This can be done by using addition. We see that the expression

$$-2(x - 3)$$

is a real number for all values of the variable x. Since we are going to use addition and not multiplication we shall not be bothered by the fact that the expression becomes equal to zero for $x = 3$. Adding our expression to both sides we obtain

$$x(x - 3) + \big(-2(x - 3)\big) = 2(x - 3) + \big(-2(x - 3)\big) ,$$

which becomes

$$x(x - 3)-2(x - 3) = 0.$$

Using the distributive law we can rewrite our equation as

$$(x - 2)(x - 3) = 0.$$

Our equivalent sentence is the compound sentence

$$x - 2 = 0 \quad \text{or} \quad x - 3 = 0.$$

This compound sentence has the truth set $\{3, 2\}$. Thus we see that the sentences "$x(x - 3) = 2(x - 3)$" and "$x - 2 = 0$ or $x - 3 = 0$" are equivalent.

Check Your Reading

1. What is the domain of the variable in "$x(x - 3) = 2(x - 3)$"?
2. Multiplication of both sides of "$x(x - 3) = 2(x - 3)$" by $\frac{1}{x - 3}$ results in the loss of what element of the truth set?
3. Why is "$x(x - 3) - 2(x - 3) = 0$" equivalent to "$x(x - 3) = 2(x - 3)$"?
4. What property is used to factor "$x(x - 3) - 2(x - 3)$", to obtain "$(x - 2)(x - 3)$"?

Oral Exercises 15-2b

In each of the following, give another sentence which is equivalent to the given sentence.

1. $x^2 = x$
2. $x^2 - x = 0$
3. $2m = m^2$
4. $2m - m^2 = 0$
5. $y^2 = 5y^2$
6. $z(z - 1) = 2(z - 1)$
7. $t(t + 1) - 3(t + 1) = 0$
8. $(t + 2)^2 - 2(t + 2) = 0$
9. $x(1 - x) - 3(1 - x) = 0$
10. $(1 - y)^2 = 5(1 - y)$

Problem Set 15-2b

1. Find the truth sets of:
   (a) $x^2 - 3x = 0$
   (b) $y^2 + 5y = 0$
   (c) $t^2 = 4t$
   (d) $m^2 = -9m$
   (e) $t(t - 1) = 2(t - 1)$
   (f) $z(z + 1) = 5(z + 1)$

Problem Set 15-2b
(continued)

2. Solve:

(a) $\frac{1}{4}m^2 = m$

(b) $\frac{x^2 - 5}{x - 1} = 5$

(c) $4(2y - 1) = y(2y - 1)$

(d) $t^2 - 1 = 6t - 6$

(e) $3x = 5x^2$

(f) $2x + \frac{x + 4}{x + 1} = 4$

3. Solve:

(a) $2x^2 + 3(x + 1) = x(x + 8) + 3$

(b) $5y(y + 1) = 3y(y + 2)$

(c) $\frac{m}{m - 2} = 4m$

(d) $\frac{1}{t + 1} - \frac{1}{t - 1} = 3$

(e) $t(t^2 + 1) = 2(t^2 + 1)$

(f) $(y - 8)(y + 3) = -30$

4. Solve:

(a) $\frac{x}{3} + \frac{3}{x} = \frac{37}{6}$

(b) $\frac{x}{x - 1} = \frac{2}{x + 5}$

(c) $3 = y - \frac{y - 3}{y + 1}$

(d) $2m^2 + 5m = 12$

(e) $z - \frac{1}{z + 3} = \frac{z^2 + 5}{z + 3}$

(f) $t(t + 3)(2t - 1) = 0$

5. The square of a number is four times the number itself. Write and solve an open sentence to find the numbers for which this is true.

6. The product of a positive integer and its successor (the successor to a positive integer n is $n + 1$) is six times the successor. Find the integer for which this statement is true by writing and solving an open sentence.

15-3. Squaring.

If we begin with the sentence

$$x = 5,$$

and square both sides, we get the sentence

$$x^2 = 25.$$

Is this second sentence equivalent to the first one? The second sentence has the truth set $\{5, -5\}$. Do you see why this is so? (Remember $(-5)\cdot(-5) = 25$, and $5 \cdot 5 = 25$.)

However, the first sentence has a truth set with only one element. Its truth set is $\{5\}$.

It should be clear, then, that the operation of squaring both sides will not necessarily produce an equivalent sentence. Will truth numbers be "gained" or "lost"?

We do know one thing. If there is a value of x which will make the sentence $x = 5$ true, then this same value of x will make the sentence $x^2 = 25$ true also. Why? What does this tell us? It tells us that the new truth set will contain the elements of the old one. It may, however, contain some new elements as well. In other words, when both sides of a sentence are squared no truth numbers are lost but new elements may be gained which do not satisfy the original sentence.

There are some types of sentences which we can best solve by squaring both sides. In such cases we may obtain sentences which are not equivalent. The truth sets of the new sentences may contain "extra" elements. We will need to check carefully to see which elements in the truth set of the new sentence are also elements of the truth set of the original sentence. Here are some examples.

Example 1: Solve the equation $\sqrt{x + 3} = 1$.
If $\sqrt{x + 3} = 1$ is true for some value of x,
then $(\sqrt{x + 3})^2 = (1)^2$ is true for the same x.
We square both sides and obtain $x + 3 = 1$.
The truth set of this last sentence is $\{-2\}$.
We need now to find out whether or not -2 is a truth value of the sentence

$$\sqrt{x + 3} = 1.$$

If $x = -2$ then the sentence reads

$$\sqrt{(-2) + 3} = 1.$$

This is true, since

$$\sqrt{1} = 1.$$

Thus, we see that -2 is the solution to the sentence "$\sqrt{x + 3} = 1$."
In this case no new element was added to the original truth set by squaring.

Example 2: Solve the open sentence $\sqrt{x} + x = 2$.
We want to obtain an equation which does not contain radicals.
We try to accomplish this by squaring both sides.
This gives us

$$(\sqrt{x} + x)^2 = 2^2$$

which becomes

$$(\sqrt{x})^2 + 2(\sqrt{x})(x) + x^2 = 2^2$$

and this can be written

$$x + 2x\sqrt{x} + x^2 = 4$$

The sentence still contains a radical. This suggests that we should begin with a different step.
Beginning again with "$\sqrt{x} + x = 2$,"
let's first add $(-x)$ to both sides. This will give us

$$\sqrt{x} = 2 - x$$

which is equivalent to the original sentence. Do you see why?
If we square both sides, we will get

$$(\sqrt{x})^2 = (2 - x)^2$$

which becomes

$$x = 4 - 4x + x^2.$$

Adding $-x$ to both sides we get

$$0 = 4 - 5x + x^2$$

which becomes

$$0 = (x - 4)(x - 1).$$

This sentence has the truth set $\{4, 1\}$.
We know that if there are any elements in the truth set of our original sentence, then these will be contained in the set $\{4, 1\}$.
Once again we must check. We see that 1 is a truth value since

$$1 + 1 = 2 \text{ is true.}$$

But what about the number 4? We see that the sentence

$$\sqrt{4} + 4 = 2 \text{ is clearly false.}$$

Therefore, we know that the truth set of our original sentence is

$$\{1\}.$$

Evidently 4 is an "extra" element which has come in because of the squaring process.

Check Your Reading

1. Are "$x = 5$" and "$x^2 = 25$" equivalent open sentences?
2. Which of the following two statements is true?

   Every truth number of "$x = 5$" is also a truth number of "$x^2 = 25$."

   Every truth number of "$x^2 = 25$" is also a truth number of "$x = 5$."
3. If both sides of an open sentence are squared, is the resulting sentence necessarily equivalent to the original one?
4. Are 4 and 1 both truth numbers of "$\sqrt{x} + x = 2$"?

Oral Exercises 15-3

Solve each of the following by squaring.

1. $\sqrt{x} = 3$
2. $\sqrt{y} + 1 = 2$
3. $\sqrt{m - 1} = 2$
4. $\sqrt{z - 1} = 3$
5. $5 = \sqrt{x - 25}$
6. $5 = \sqrt{x} - 2$
7. $3 = \sqrt{y + 3}$
8. $2\sqrt{t} = 1$
9. $\sqrt{2x} = 1$
10. $2\sqrt{2x} = 1$

Problem Set 15-3

1. Solve the following sentences by squaring both sides:
    - (a) $\sqrt{y} = 3$
    - (b) $\sqrt{2x} = 4$
    - (c) $\sqrt{2m - 1} = m$
    - (d) $\sqrt{t + 12} = t$
    - (e) $z = \sqrt{6 - z}$
    - (f) $\sqrt{2x} = 1 + x$
2. Solve:
    - (a) $2\sqrt{m} = 4$
    - (b) $t = \sqrt{2t + 15}$
    - (c) $\frac{1}{2} = \sqrt{3x}$
    - (d) $\sqrt{y^2 - 1} = y$
    - (e) $2\sqrt{z} = 1 + z$
    - (f) $\sqrt{y - 2} = 9$
3. Solve:
    - (a) $\sqrt{4m} = m - 3$
    - (b) $t = \sqrt{t + 1} - 1$
    - (c) $x + 1 = \sqrt{2x + 1}$
    - (d) $|x| = 2x - 1$ (Remember: $|x|^2 = x^2$.)
    - *(e) $|y - 2| = 3$
    - *(f) $|2t| = t + 1$
4. Solve:
    - (a) $2t = \sqrt{3 - t}$
    - (b) $\sqrt{y^2 + 7} = 4$
    - (c) $\sqrt{m(m + 3)} - 1 = m$
    - (d) $\sqrt{x} + 3 = 2$
    - *(e) $|x| + 2 = 3x$
    - *(f) $t - |t| = 1$

Problem Set 15-3
(continued)

5. The square root of a certain number is twelve less than the number itself. Find the number or numbers for which this is true by writing and solving an open sentence for this problem.

6. The sum of a number and the absolute value of that number is 8. Write and solve an open sentence to find the number or numbers for which this statement is true.

7. The sum of a number and the absolute value of the number is 0. Write and solve an open sentence to find the numbers for which this is true.

15-4. Equivalent Inequalities.

Up to now in this chapter we have been studying open sentences involving the equality symbol "=" which tells us that the two sides name the same number. We have referred to these sentences as equations. In finding truth sets to these equations it has been very helpful to form equivalent sentences.

We will now find the truth sets of, or solve, open sentences such as

$$2x + 5 > 23 \quad \text{and}$$

$$3x - 4 < 20.$$

You will recall that the symbol ">" means "is greater than", and the symbol "<" means "is less than". We shall also use combined symbols such as "≥", which means "is greater than or equal to".

In this work, as in solving equations, we will want to form equivalent sentences. We therefore need some ways of doing this. The rules are very much like the rules for working with equations. However, we must be careful to note that there is one important difference. To recall this, we must return to the properties of order which we studied in Chapter 9. These properties, and the fact that certain operations can be "reversed," will give us rules for forming equivalent sentences.

The

addition property of order

states that if a, b, and c are real numbers and if $a < b$, then

$$a + c < b + c.$$

A similar property holds for the order relation ">", "is greater than". Because addition by a real number can be "reversed", we can be sure that if we add a real number to both sides of an inequality, we will get an equivalent inequality. Consider, for example, the open sentence

$$x + 5 < 8.$$

We add -5 to both sides and obtain an equivalent sentence

$$x + 5 + (-5) < 8 + (-5)$$

which becomes

$$x < 3.$$

The truth set for this last inequality is the set of all real numbers less than 3. This is also the truth set of the original sentence.

The

multiplication property of order

is a bit more complicated and needs to be reviewed carefully. In this case it makes an important difference whether we multiply by a positive real number or by a negative real number. The property can be stated as follows:

For any real numbers a and b such that $a < b$,

$(c)(a) < (c)(b)$ if c is positive

but

$(c)(b) < (c)(a)$ if c is negative.

A similar property holds for the order relation ">", "is greater than". This can be summed up by saying, as we did in Chapter 9, that if two different numbers are each multiplied by the same positive number, then the order relation remains the same. On the other hand, if both numbers are multiplied by the same negative number, then the order is reversed. Multiplication by non-zero real numbers is reversible. We can therefore apply these ideas in forming equivalent inequalities. For example, consider the inequality

$$3x < 15.$$

If we multiply both sides by the positive number $\frac{1}{3}$, we will obtain an equivalent inequality

$$x < 5.$$

Its truth set is the set of all real numbers less than 5. This is also the truth set of the original sentence.

In the sentence

$$2x + 5 > 23$$

we can add (-5) to both sides. We may then multiply both sides by the positive number $\frac{1}{2}$. This will give us an equivalent sentence

$$x > 9$$

whose truth set is the set of all real numbers greater than 9.

We will conclude with two more examples.

Example 1. Solve $\frac{3}{5}y - 2 < \frac{1}{3}y + \frac{2}{3}$

We may first multiply both sides by the positive number 15 to get an equivalent sentence without fractions. This is

$$9y - 30 < 5y + 10.$$

Now we add the real number $30 - 5y$ to both sides. This gives us

$$4y < 40.$$

Multiplication by the positive real number $\frac{1}{4}$ then gives us

$$y < 10.$$

The set of all real numbers less than 10 is the truth set of the original sentence.

Example 2. Solve $\frac{1}{-(x^2+1)} > -1$.

$-(x^2 + 1)$ represents a negative number for all values of x. Therefore, multiplying both sides of the above sentence by $-(x^2 + 1)$, we get the equivalent sentence

$$-(x^2+1)(-1) > -(x^2+1)\left(\frac{1}{-(x^2+1)}\right)$$

$$x^2 + 1 > 1$$

Notice that the order was reversed since both sides were multiplied by a negative number.

Now we add -1 to both sides and get

$$x^2 > 0.$$

The truth set of this final sentence is the set of all non-zero numbers. This is also the truth set of the original inequality.

Check Your Reading

1. Give the meanings of the following symbols

   $<, \quad >, \quad \leq, \quad \geq.$

2. State the operations which may be performed on an inequality without changing the truth set or the order.

3. What operation changes the order of an inequality?

Oral Exercises 15-4

Describe the truth sets of the following inequalities.

1. (a) $x > 5$
   (b) $y < 4$
   (c) $w \geq 10$
   (d) $z \leq -6$
   (e) $x + 1 > 7$
   (f) $y - 1 \leq \frac{3}{4}$
   (g) $2y < 6$
   (h) $-3x < -12$
   (i) $-y \leq -4$
   (j) $-x + 1 < 8$

Problem Set 15-4

1. Solve the following inequalities by writing equivalent inequalities.
   (a) $x + 12 < 39$
   (b) $t - 7 > 24$
   (c) $y + 7 < -2$
   (d) $x + 3 < 2x - 8$
   (e) $3x + 6 > 12$
   (f) $5x - 2 > x + 10$

2. Solve:
   (a) $8y - 3 > 3y + 7$
   (b) $\frac{1}{2}x + \frac{5}{2} < \frac{7}{2}$
   (c) $-3x > 6$
   (d) $t + 5 < 2t + 12$
   (e) $\frac{x}{2} > \frac{1}{3}$
   (f) $\frac{y}{3} + \frac{5}{2} < \frac{y}{6} + 1$

3. Solve:
   (a) $\frac{t}{3} < 4 + \frac{t}{6} - 2$
   (b) $\frac{m}{\sqrt{m}} < 3$
   (c) $\frac{x}{-3} > 2$
   (d) $\frac{3}{x^2 + 4} < -2$
   (e) $-t\sqrt{2} < \sqrt{2}$
   (f) $\frac{1}{x} > \frac{1}{3}$ and $x > 0$

4. Solve:
   (a) $\frac{4}{t} < 5$ and $t > 0$
   (b) $x^2 + 2x > x^2 + 4$
   (c) $(x - 2)(x - 3) > (x + 2)(x - 5)$
   (d) $(y + 1)(y + 2) < (y + 3)(y + 4)$

Problem Set 15-4
(continued)

(e) $\frac{1}{y} < -\frac{1}{2}$ and $y < 0$

(f) $\frac{3}{m} > 2$ and $m < 0$

Summary

The problem of finding truth sets for open sentences involves the following important ideas.

1. Open sentences which have the same truth sets are called equivalent.
2. Sentences with the connecting symbol "=" are called equations.
3. Two equations are equivalent if one can be obtained from the other by
    (a) Addition of the same real number to both sides.
    (b) Multiplication of both sides by the same non-zero real number.
4. The following operations will not always produce equivalent equations.
    (a) Multiplication of both sides by an expression containing a variable if there is an element in the domain of the variable for which this expression does not represent a real number.
    (b) Adding to both sides an expression containing a variable if there is an element in the domain of the variable for which this expression does not represent a real number.
    (c) Multiplication of both sides by an expression containing a variable if this expression can have the value zero for some element in the domain of the variable.
    (d) Squaring both sides.

5. Unless otherwise specified, the domain of the variable of any given sentence will be taken as the set of all real numbers for which the sentence has a meaning.
6. A sentence in factored form of the type $(x - 3)(x - 4) = 0$, is equivalent to the compound sentence $x - 3 = 0$ or $x - 4 = 0$.
7. Two inequalities are equivalent if the second can be obtained from the first by adding the same number to both sides of the first.
8. Two inequalities are equivalent if the second can be obtained from the first by multiplying both sides of the first by the same positive real number.
9. Two inequalities are equivalent if one of them is obtained from the other one by multiplication of both sides of the first by the same negative real number and reversing the order.

Review Problem Set

Find the truth set of:

1. $x - 7 = 8x$
2. $\frac{y}{5} - \frac{1}{3} = \frac{2y + 2}{15}$
3. $8x + 2 = 3x + \frac{9}{2}$
4. $\sqrt{3x} = 3$
5. $z - \frac{1}{z} = \frac{8}{3}$
6. $t^2 = 4t$
7. $|x| + 6 = 2|x|$
8. $\frac{1}{2}x + 8 = \frac{1}{3}x$
9. $z(z - 1) = 0$
10. $z(z - 1) = 6$
11. $2m + 4 > m + 6$
12. $\frac{1}{t} = 5$
13. $\frac{1}{t} < 5$ and $t > 0$
14. $17x + 24 = 2x - 21$
15. $\sqrt{y + 1} = 6$
16. $\frac{n}{n - 3} - \frac{3}{n - 3} = 0$
17. $\frac{n}{n - 3} + \frac{3}{n - 3} = 0$
18. $\frac{1}{x} - \frac{4}{x - 1} = 1$
19. $(x - 2)(x - 3) < (x - 4)(x - 5)$
20. $\frac{y + 2}{y - 2} = 0$
21. $m(m - 1) = 4(m - 1)$
22. $\frac{t}{t + 1} = \frac{t - 1}{t + 3}$
23. $\frac{x^2}{x^2 + 1} < 1$
24. $-3m < 6$

Review Problem Set
(continued)

25. $x^2 + 11x + 24 = 0$

26. $y = \sqrt{5y + 24}$

27. $6t^2 + 5 = 5(t^2 + 1) + 2t$

28. $\frac{2}{3}x > 8$

29. $\frac{20}{z^2 + 1} = 2$

30. $\frac{2}{q - 8} = \frac{1}{q - 2}$

31. $\frac{1}{t} - 3 + \frac{2t - 1}{t} = 0$

32. $|x| + x = 0$

33. $\frac{m + 1}{m + 1} = 1$

34. $\sqrt{y} + y = 2$

35. The numerator of a fraction is 7 less than the denominator. The value of the fraction is $\frac{2}{3}$. What is the numerator? What is the denominator?

36. A salesman made a trip of 150 miles at a certain average rate. The next day by increasing his rate 10 miles per hour he made a 200 mile trip in the same time as the previous trip. What was his average rate on the first trip?

37. One third of a number is equal to $\frac{2}{5}$ of 3 more than the number. Find the number.

38. Is there an integer such that the result of dividing 4 by five less than the integer is the same as the result of dividing 5 by four less than the integer?

39. Bill can mow the lawn at his father's country cottage in 2 hours when he uses a power mower. His brother can mow the same lawn in five hours with a 'push' mower. If the boys work together, how long will it take?

Review Problem Set
(continued)

40. When 3 is divided by 2 more than a certain number the result is the same as 7 divided by that number. What is the number?

41. The sum of seven divided by a certain number and twelve divided by the square of the same number is -1. What is the number?

42. There are 90 students in a high school band. They are arranged on a football field so that the number of rows is 9 less than the number of students in each row. How many rows are there?

43. The sum of a certain number and its absolute value is greater than 6. What is the number?

44. The reciprocal of a certain number equals the number multiplied by the reciprocal of 36. What is the number?

Draw the graph of the truth set of each of these sentences on a number line.

45. $x + 1 > 4$ and $2x = 8$

46. $|x| + 2 = 7$

47. $x(x + 3) = 0$

48. $x - \frac{6}{x} = 1$

49. $\sqrt{x + 1} = 4$

50. $-5x < 10$

51. $\frac{s}{s - 5} + \frac{5}{s - 5} = 0$

CHALLENGE PROBLEMS

1. A farmer has \$10,000 to buy steers at \$250 each and cows at \$260 each. If you know the number of steers and the number of cows are each positive integers, what is the greatest number of animals he can buy, if he must use the entire \$10,000?

2. At what time between 3 and 4 o'clock will the hands of a clock be together? At what time will they be opposite each other?

3. A rug with area of 24 square yards is placed in a room 14 feet by 20 feet leaving a uniform width around the rug. How wide is the strip around the rug? A sketched diagram of the rug upon the floor may help you represent algebraically the length and width of the rug.

4. One leg of a right triangle is 2 feet more than twice the smaller leg. The hypotenuse is 13 feet. What are the lengths of the legs?

5. Tell which of these numbers are rational:

   $\sqrt[3]{\pi^3}$, $\sqrt{.4}$, $\sqrt[3]{.0008}$, $(\sqrt[3]{-1})(\sqrt{.16})$.

6. If a two-digit number of the form $10t + u$ is divided by the sum of its digits, the quotient is 4 and the remainder is 3. Find the numbers for which this is true.

7. Simplify $1 + \dfrac{1}{1 + \dfrac{1}{1 + \frac{1}{2}}}$.

8. (a) For what positive integral values of k is the polynomial $x^2 + kx + 12$ factorable over the integers?

   (b) For what positive integral values of k is the polynomial $x^2 + 6x + k$ factorable over the integers?

   (c) Determine the value of k so that $x^2 - 6\sqrt{3}x + k$ is a perfect square.

9. For each integer n show whether or not the integer $(n + 3)^2 - n^2$ is divisible by 3.

10. Show whether or not $x - 3$ is a factor of the polynomial $x^4 - 5x^3 + 6x^2 - 3$.

11. Suppose that the polynomial $5x^{100} + 3x^{17} - 1$ is written in the form

$$5x^{100} + 3x^{17} - 1 = q(x^3 - x^2 + 1) + r ,$$

where q and r are polynomials.

   (a) What can you say about the degree of r if it is as low as possible?

   (b) If r has minimum degree, what is the degree of q?

12. The polynomial $2x^4 + 1$ can be written in the form

$$2x^4 + 1 = 2(x^3 + x^2 + x + 1)(x - 1) + r$$

where r is an integer. If a value of x is given, can you find r without carrying out the division process? Is there a special value of x for which this is easiest?

13. The polynomial $5x^{100} + 3x^{17} - 1$ can be written in the form

$$5x^{100} + 3x^{17} - 1 = q(x - 1) + r ,$$

where q is a polynomial and r is an integer. Find the integer r without carrying out the division process.

14. A man makes a trip of d miles at an average speed of r miles per hour and returns at an average speed of q miles per hour. What was his average speed for the entire trip?

15. Prove the

Theorem: If a and b are distinct positive real numbers, then

$$\frac{a + b}{2} > \sqrt{ab}.$$

Hint: Observe that proving $\frac{a + b}{2} > \sqrt{ab}$ is equivalent to proving $a + b - 2\sqrt{ab} > 0$.

16. Solve $x^4 = 1$ by thinking of x^4 as $(x^2)^2$ and then factoring.

17. A remarkable expression for producing prime numbers is

$$n^2 - n + 41.$$

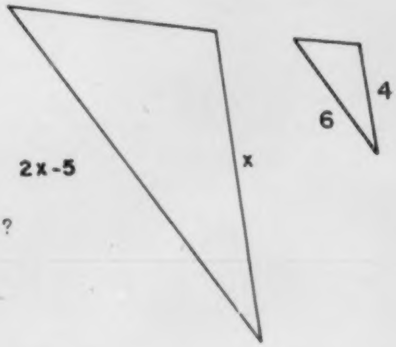
Try it for several integral values of n from 0 to 40. Is the result in every case a prime number? Try the number 41. Is the result a prime number? If an open sentence is true for 400 values of the variable does it necessarily follow that the 401st value chosen will also make it true?

18. Prove: If $a > 0$, $b > 0$ and $a > b$, then $\sqrt{a} > \sqrt{b}$. Hint: Use the comparison property in an indirect proof. That is, assume $\sqrt{a} \leq \sqrt{b}$ and see if this leads to some contradiction of other conditions we have set up, or of properties we have agreed upon.

19. A rat which weighs x grams is fed a rich diet and gains 25% in weight. He is then put on a poor diet and loses 25% of his weight. Find the number of grams difference in the weight of the rat from the beginning of the experiment to the end.

20. A man needs 7 gallons of paint to paint his house. He bought three times as much grey paint at \$6 a gallon as white paint at \$7 a gallon. How many gallons of each color paint did he buy? (Assume that paint can be purchased in quart size as well as gallon size cans.)

21. The sides of lengths x and $2x - 5$ of the first triangle shown have the same ratio as the sides of lengths 4 and 6 respectively of the second triangle. How long are the two sides of the first triangle?



22. A rectangular bin is 2 feet deep and its perimeter is 24 feet. If the volume of the bin is 70 cu. ft., what are the length and the width of the bin?

23. Two plywood panels each of which cost 30¢ per square foot, were found to have the same area although one of them was a square and the other was a rectangle 6 feet longer than the square but only 3 feet wide. What were the dimensions of the two panels?

24. Prove that if p and q are integers and if $x^2 + px + q$ is factorable, then $x^2 - px + q$ is also factorable.

25. Find every integer p such that $x^2 + px + 36$ is factorable. For which values of p will $x^2 + px + 36$ be a perfect square? How are these values of p distinguished from the other values? Answer the same questions for the polynomial $x^2 + px + 64$. If n is a positive integer, what is your guess as to the smallest positive integer p for which $x^2 + px + n^2$ is factorable? What is your guess as to the largest positive integer p for which $x^2 + px + n^2$ is factorable?

26. A mixture for killing weeds must be made in the ratio of 3 parts of weed-killer to 17 parts of water. How many quarts of weed-killer should be put in a 10 gallon tank which is going to be filled up with water to make 10 gallons of mixture?

27. Factor each of the following as shown by the example.

Example:

$$\begin{aligned} a^3 + b^3 &= a^3 - ab^2 + ab^2 + b^3 \\ &= a(a^2 - b^2) + (a + b)b^2 \\ &= a(a + b)(a - b) + (a + b)b^2 \\ &= (a + b)(a(a - b) + b^2) \\ &= (a + b)(a^2 - ab + b^2) \end{aligned}$$

(a) $t^3 + 1$

(b) $s^3 + 8$

(c) $27x^3 + 1$

28. Factor each of the following as shown by the example.

Example: $a^3 - b^3 = a^3 - ab^2 + ab^2 - b^3$

$= a(a^2 - b^2) + (a - b)b^2$

$= a(a - b)(a + b) + (a - b)b^2$

$= (a - b)(a(a + b) + b^2)$

$= (a - b)(a^2 + ab + b^2)$

(a) $t^3 - 1$

(b) $s^3 - 8$

(c) $8x^3 - 1$

GLOSSARY CHAPTERS 11-15

DEGREE OF A POLYNOMIAL - The degree of a polynomial in one variable is given by its term of highest degree.

DIVISION PROCESS - For any two polynomials n and d, where d is not the zero polynomial, there exist polynomials q and r, with r of lower degree than d such that

$$n = qd + r$$

and

$$\frac{n}{d} = q + \frac{r}{d}.$$

DOMAIN OF THE VARIABLE - The domain of the variable is the set of numbers from which the value of the variable may be chosen.

EQUATIONS - Open sentences involving the equality symbol ("=", which tells us that the two sides name the same number) are referred to as equations.

EQUIVALENT SENTENCES - Open sentences which have the same truth set are called equivalent sentences.

EXPONENT - In the expression n^a, a is the exponent. It shows how many times the base n is to be used as a factor.

FACTOR - We say a positive integer x is a factor of a positive integer y if y is a multiple of x.

IRRATIONAL NUMBER - In the set of real numbers, a number which is not rational is called an irrational number.

LEAST COMMON MULTIPLE - The least common multiple of two integers is the smallest positive integer that has each of them as a factor.

MONOMIAL - A monomial is a polynomial in which the only indicated operations are multiplication, taking the opposites, or in which no operation is indicated.

POLYNOMIAL - A polynomial is any numeral for an element in a set of numbers, or any variable, or any expression which indicates operations of addition, subtraction, multiplication or taking the opposites of any elements in the set and of the variables.

POLYNOMIAL EQUATION - A polynomial equation is an equation in which each side is a polynomial.

PRIME FACTORIZATION - Every integer greater than one can be written as an indicated product in which every factor is a prime number. Such an indicated product is called the prime factorization of the integer.

PRIME NUMBER - A prime number is an integer greater than one which has no proper factors.

PRIME POLYNOMIAL - If a polynomial cannot be written as the product of two polynomials it is said to be a prime polynomial.

PROPER FACTORS - In the set of positive integers, the proper factors of a number are all its factors except the number itself and one.

QUADRATIC POLYNOMIAL - A polynomial of the second degree in one variable is known as a quadratic polynomial.

RATIONAL EXPRESSIONS - Expressions which involve numbers and variables and indicate at most the operations of addition, subtraction, multiplication, division and taking opposites.

RATIONAL NUMBERS - Numbers which can be expressed as the quotient of two integers (except for division by zero).

Chapter 16

TRUTH SETS AND GRAPHS OF SENTENCES IN TWO VARIABLES

16-1. Open Sentences in two Variables.

There have probably been occasions when you felt that it would have been convenient to express the conditions of a problem in the form of a sentence in two variables. We shall now consider sentences such as

$$2x + y = 5.$$

We say this sentence is of first degree because the phrases involved are of first degree or less. It is also called an equation of the first degree in two variables.

If the domain of x and the domain of y are both the set of real numbers, then we may consider x to be any real number and do the same for y. For example, we might choose 3 as the value for x, and 7 as the value for y. Then the sentence becomes

$$2(3) + 7 = 5;$$

clearly this is a false sentence.

Can you select from the set of real numbers a value for x and a value for y such that the sentence is true? Do you believe that, if the same question were asked of three different people, the values offered for x and y would be the same in each case?

Your selection may have been $x = 2$ and $y = 1$. With these values of x and y, the open sentence

$$2x + y = 5$$

becomes

$$2(2) + 1 = 5,$$

which is a true sentence.

There are other possible choices which make the sentence true. Some are shown in the following table. Notice that the choices appear in pairs.

	value for x	corresponding value for y
one pair	$\frac{1}{2}$	4
a second pair	3	-1
a third pair	0	5

Can you find several more pairs which make the sentence true?

As we can see, our truth set has many elements in it, each of which is a pair of real numbers. As a matter of fact we can choose any real number as a value for x and then find a corresponding real number as a value of y so that for these two numbers the sentence will be true. Can you answer the question "How many pairs of values can be chosen for x and y so that "$2x + y = 5$" is a true sentence?"

The truth set of a sentence in two variables contains infinitely many elements, each of which is a pair of real numbers.

It is inconvenient to have to say always "x is 2 and y is 1" when we are describing these number pairs. We agree to a simple symbol which says the same thing, "(2, 1)", which means "x is 2 and y is 1". If we wished to say "x is 1 and y is 2" we would use the symbol "(1, 2)". In other words, the numeral on the left represents the value of x and the numeral on the right represents the value of y. Thus, "(2, 1)" and "(1, 2)" are not the same pair. You should also notice that the first pair makes the sentence true while the second does not. Verify this.

Because the value of x is given first, followed by the value of y, the order of the numbers is important. The symbol "(2, 1)" is said to represent an "ordered pair of numbers".

An ordered pair belonging to the truth set of a sentence with two variables is called a solution of the sentence, and this ordered pair is said to satisfy the sentence. We also speak of an ordered pair being a solution of an equation, or satisfying an equation.

Check Your Reading

1. Why is "$2x + y = 5$" a polynomial equation of the first degree?
2. Complete the sentence: Each element of the truth set of "$2x + y = 5$" is a ________ of real numbers.
3. How many elements are in the truth set of "$2x + y = 5$"?
4. Complete the sentence: The number pair $(2, 1)$ means that x is _____ and y is _____.
5. Is $(2, 1)$ the same number pair as $(1, 2)$?
6. Why do we call $(2, 1)$ an ordered pair of numbers?
7. Complete the sentence: An ordered pair belonging to the truth set of a sentence with two variables is called a ______________ of the sentence, and this ordered pair is said to ______________ the sentence.

Oral Exercises 16-1a

1. Name several ordered pairs which represent values of x and y that make the sentence $x + 2y = 5$ true. State some ordered pairs which make it false.
2. What does "$(5, 1)$" mean? What does "$(-2, 7)$" mean?
3. Name any element of the set of ordered pairs of real numbers and tell what it means. How many numerals are required to name one element of this set?
4. When y and x represent elements of an ordered pair, which have we agreed to list first?
5. Why is it impossible to list all the elements of the truth set of a sentence of first degree in two variables?
6. How many numbers are there in one element of the truth set of an open sentence in one variable?
7. How many numbers are in one element of the truth set of an open sentence in two variables?

Oral Exercises 16-1a
(continued)

8. Why do we use an ordered pair to list an element of the truth set of an open sentence in two variables?

9. Consider the sentence $x + y = 5$. Which of the following ordered pairs are solutions of the sentence?

   $(-3, 8)$ $(-1, 6)$ $(0, 5)$ $(1, 4)$ $(6, -1)$ $(7, -2)$

10. How many elements are in the truth set of the open sentence in two variables given in Problem 9?

Problem Set 16-1a

1. Given the open sentence $2x + y = 8$. Which of the following ordered pairs are elements of the truth set of the sentence? Which are not?

   (a) $(3, 2)$
   (b) $(-1, 10)$
   (c) $(4, 4)$
   (d) $(6, -4)$
   (e) $(0, 6)$
   (f) $(8, -8)$
   (g) $(-5, 2)$
   (h) $(0, 8)$
   (i) $(1, 6)$
   (j) $(-4, 8)$

2. Complete the following ordered pairs such that each satisfies the open sentence $x + y = 6$.

   (a) (5, ___)
   (b) (6, ___)
   (c) (0, ___)
   (d) (-1, ___)
   (e) (___, 8)
   (f) (___, 3)
   (g) (___, -14)
   (h) (-9, ___)
   (i) (-6, ___)
   (j) (-7, ___)

Problem Set 16-1a
(continued)

3. Which of the following sentences are true for the given ordered pair? Which are false?

    (a) $2x + 2y = 8$; $(3, 1)$
    (b) $2x + y = 0$; $(1, -2)$
    (c) $x + 5y = -10$; $(4, -2)$
    (d) $y - 3x = 0$; $(3, 1)$
    (e) $4x - 2y = -14$; $(-2, 3)$
    (f) $2x - 3y = 6$; $(2, 3)$
    (g) $5x + 2y = 8$; $(0, 4)$
    (h) $x = 2y + 3$; $(7, 2)$
    (i) $x - 5y - 2 = 0$; $(7, 1)$
    (j) $2x - y + 4 = 0$; $(2, 2)$
    (k) $y - 6x - 2 = 0$; $(1, 8)$

4. For each of the following sentences list three elements of the truth set.

    (a) $x + y = 3$
    (b) $x + y = -2$
    (c) $x - y = 7$
    (d) $2x - y = 4$
    (e) $7x + y = 9$
    (f) $\frac{1}{2}x + y = 6$
    (g) $3x + y = \frac{3}{4}$
    (h) $2x - 4y = 9$
    (i) $x - y = \frac{1}{9}$
    (j) $11x + 7y = 18$

5. The sum of the digits of a two digit number is 11. What is the number?

    (a) Write a sentence in two variables to describe this problem.
    (b) State the domains of each of the two variables.
    (c) Give the truth set. (There are eight elements.)

If we wish to find some elements of the truth set of a sentence such as

$$4x + 2y - 5 = 0,$$

we might find it more convenient to write an equivalent sentence with y by itself on the left side. When we say "by itself" we mean, of course, with coefficient 1. This can be done in the following manner.

$$4x + 2y - 5 = 0$$

is equivalent to

$$4x + 2y = 5,$$

which is equivalent to

$$2y = -4x + 5.$$

(What operation has been performed?)

This is equivalent to

$$y = -2x + \frac{5}{2}.$$

(What operation has been performed?)

This last sentence

$$y = -2x + \frac{5}{2}$$

is called the "y-form" of the sentence $4x + 2y - 5 = 0$.

In general, every first degree sentence in two variables where the coefficient of y is not zero can be written in the "y-form"

$$y = ax + b,$$

where a and b are specific real numbers.

We shall see that this form reveals important information about the relation between the values of x and y for which the sentence is true. It also makes it possible for us to determine quite easily certain elements of the truth set of the sentence.

Returning to our example,

$$y = -2x + \frac{5}{2},$$

some elements of its truth set can now be easily obtained. We need only replace x by each of several values and we can quickly find the corresponding values for y as the table shows.

If x is	then the sentence $y = -2x + \frac{5}{2}$ becomes	and the corresponding y value is	so an element of the truth set is
0	$y = -2(0) + \frac{5}{2}$	$\frac{5}{2}$	$(0, \frac{5}{2})$
1	$y = -2(1) + \frac{5}{2}$	$\frac{1}{2}$	$(1, \frac{1}{2})$
2	$y = -2(2) + \frac{5}{2}$	$-\frac{3}{2}$	$(2, -\frac{3}{2})$
-1	$y = -2(-1) + \frac{5}{2}$	$\frac{9}{2}$	$(-1, \frac{11}{2})$
-3	$y = -2(-3) + \frac{5}{2}$	$\frac{17}{2}$	$(-3, \frac{17}{2})$
$\frac{3}{2}$	$y = -2(\frac{3}{2}) + \frac{5}{2}$	$-\frac{1}{2}$	$(\frac{3}{2}, -\frac{1}{2})$

Let's look at another sentence:

$$3x - y + 1 = 0$$

Written in the y-form this becomes

$$y = 3x + 1.$$

From this we can describe the relation between x and y in words. It should be clear that for every value of x the corresponding value of y which makes the sentence true is "one more than three times as great". For example, when x is 2, y is 7; when x is 3, y is 10, etc. Thus the y-form makes it easy to construct a table. See how rapidly you can make a table giving five pairs in the truth set other than those mentioned above.

Check Your Reading

1. What is the y-form of "$4x + 2y - 5 = 0$"?
2. When a sentence is written in y-form, what is the coefficient of y?
3. Can every first degree sentence in two variables be written in y-form?
4. Describe in words the relationship between the values of the variables in the sentence "$y = 3x + 1$".

Problem Set 16-1b

1. Change each of the following sentences to y-form and then state the number pair in the truth set whose x value is 2; whose x value is 0; whose x value is -1.

(a) $x + y - 7 = 0$

(b) $3x + y = 12$

(c) $3x - y - 2 = 0$

(d) $2x + 2y - 1 = 0$

(e) $2x + 5y = 40$

(f) $6x + y + 4 = 0$

(g) $\frac{2}{3}x + y = \frac{1}{2}$

(h) $-2x - y + \frac{4}{3} = 0$

(i) $4x - y = \frac{5}{8}$

(j) $11x + 12y - 7 = 0$

Problem Set 16-1b
(continued)

2. Write each of the following sentences in y-form and list three ordered pairs in the truth set.

(a) $x + y - 4 = 0$

(b) $x - y - 7 = 0$

(c) $3x + y = 2$

(d) $11x - y + 5 = 0$

(e) $2x + 3y = 4$

(f) $4x - 2y = 9$

(g) $\frac{1}{2}x - 2y - 8 = 0$

(h) $\frac{2}{3}x + y = 2$

(i) $7x - 3y = 10$

(j) $20x + 7y - \frac{1}{2} = 0$

(k) $-6x + 2y = 3$

(l) $x + 12y - 7 = 0$

(m) $-3x - 2y + 1 = 0$

(n) $\frac{2}{3}x - y = \frac{1}{3}$

(o) $\frac{1}{2}x + 2y = \frac{12}{5}$

(p) $.7x + .2y - .3 = 0$

(q) $.2x - 1.5y = .7$

(r) $3x - \frac{2}{3}y = 4$

(s) $9x + 21y - 6 = 0$

(t) $1.5x - .3y = 6.3$

3. The sum of the digits of a two digit number is 15. For what numbers is this true? Write one sentence in two variables which describes this problem, find the truth set of the sentence, and then answer the question of the problem.

16-2. Graphs of Ordered Pairs of Real Numbers.

You will remember how convenient it has been to refer to the number line to gain more understanding of the real numbers. Some ideas about open sentences in one variable were made clearer by looking at the graphs of their truth sets. Truth sets of such sentences consist of elements which are real numbers. You recall that the graph of a set of numbers is the corresponding set of points on the number line.

We would like now to be able to construct the graphs of truth sets of sentences in two variables. The problem is a different one, since our elements are number pairs rather than single numbers. It would not be possible to associate each number pair with a single point on the number line, since each point corresponds to a single number.

It is likely, therefore, that we will need to consider a correspondence between number pairs and points in a plane. This can be done by means of two number lines in the following way:

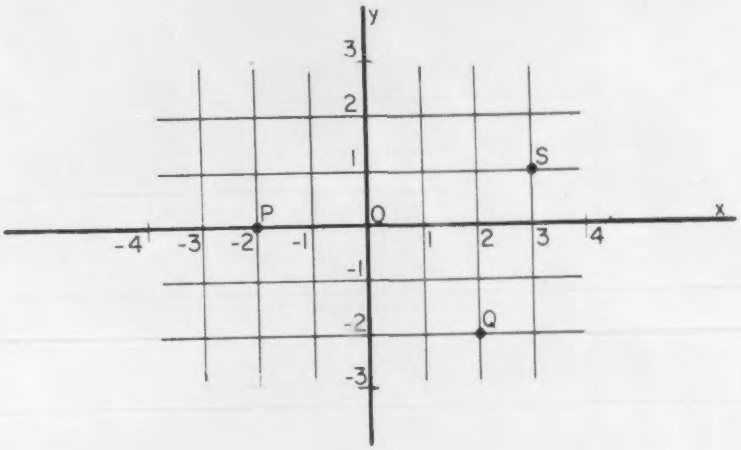


Figure 1

The horizontal line (labeled x) corresponds to our original number line. Perpendicular to it at the point O is another number line (labeled y). One unit above zero on the y-line we find the point whose coordinate is 1. One unit below zero we find the point whose coordinate is (-1). We continue to identify coordinates of the successive points on our y-line in the same manner as we did on the x-line. We put in the parallel lines forming the system of squares for convenience.

Do you see how we can use this system to "locate", for example, the point S corresponding to the number pair (3, 1)? We move three units to the right of zero, and then move up one unit.

Thus we see that the numbers of an ordered pair can be made to correspond to a point in the plane. We can think of this point as having two coordinates. In this case the coordinates are 3 and 1. The coordinate 3 is shown by a measurement of 3 units to the right of zero. It is called the *x-coordinate*. The coordinate 1 is shown by a measurement of one unit *above* zero. It is called the *y-coordinate*. Thus, an ordered pair (3, 1) corresponds to a point whose x-coordinate is 3 and whose y-coordinate is 1.

Do you agree that the ordered pair of numbers (-2, 0) is properly represented by the point P in Figure 1? The first coordinate of P tells us that P is 2 units to the *left* of 0. The second coordinate of P tells us that the distance of the point upward or downward from the x-line is zero.

Do you see that the point Q in Figure 1 corresponds to the number pair (2, -2)? The second coordinate of Q tells us that the point Q is two units below 0.

In place of "x-line" and "y-line" we shall from now on use the words "*x-axis*" and "*y-axis*". We use the plural word "axes" when we wish to refer to both of them. See Figure 2.

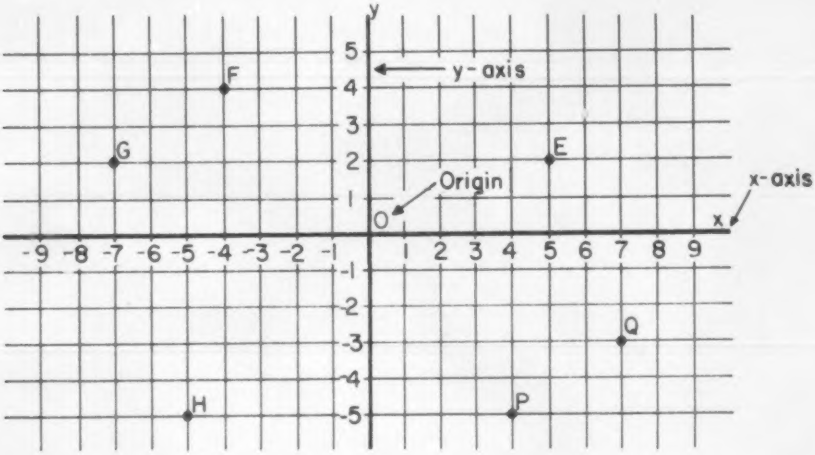


Figure 2

The intersection (crossing point) of the x-axis and the y-axis is called the origin. It is the point whose coordinates are both zero. The pair of numbers (0, 0) corresponds to the origin.

Can we associate with each ordered pair a point on the plane? For example, given any ordered pair such as (-3, 4) can you find a point having these numbers as coordinates. How would you do it? Do you see that you would locate it 3 units to the left of the origin and 4 units upward? Locating a point associated with an ordered pair of real numbers and indicating that point by placing a dot in reference to the x- and y-axes is called "plotting the point". Does every ordered pair correspond to at least one point? Exactly one point?

The answer to both questions is yes. For each ordered pair of real numbers there corresponds in the plane exactly one point. For each point of the plane there corresponds exactly one ordered pair of real numbers. This statement is very similar to the one which was made in Chapter 1 about real numbers and the number line. A correspondence such as this is said to be one-to-one.

Check Your Reading

1. How do we locate in a plane the point which corresponds to the number pair (3, 1)?
2. Complete the statement: The number pair (3, 1) corresponds to a point in a plane. The number 3 is called the _______ of the point and the number 1 is called the ________ of the point.
3. What is another name for the x-line?
4. What is another name for the y-line?
5. What do we call the point which corresponds to (0, 0)?
6. Does each number pair correspond to exactly one point?
7. Does each point correspond to exactly one number pair?
8. What do we call a correspondence such as that between number pairs and points on a plane?

Problem Set 16-2a

1. Draw a pair of axes on a page of graph paper and label them as in Figure 1.
2. On the set of axes that were drawn for Problem 1, plot each of the following points and write the coordinates as an ordered pair of numbers beside the point.

(a) (5, 0)
(b) (0, 3)
(c) (3, 5)
(d) (0, 0)
(e) (-3, 5)
(f) (4, -3)
(g) (-2, -5)
(h) (0, -4)
(i) (-4, 0)
(j) (6, -4)
(k) (6, 4)
(l) (-6, 4)
(m) (4, -6)
(n) (-4, -6)
(o) (-6, -4)

The x-coordinate of a point is usually called the *abscissa*. The y-coordinate is called the *ordinate*.

Suppose, now, we were given a point in a plane. How would we determine its coordinates? For example, what are the coordinates of the points R, M, and S on the diagram below?

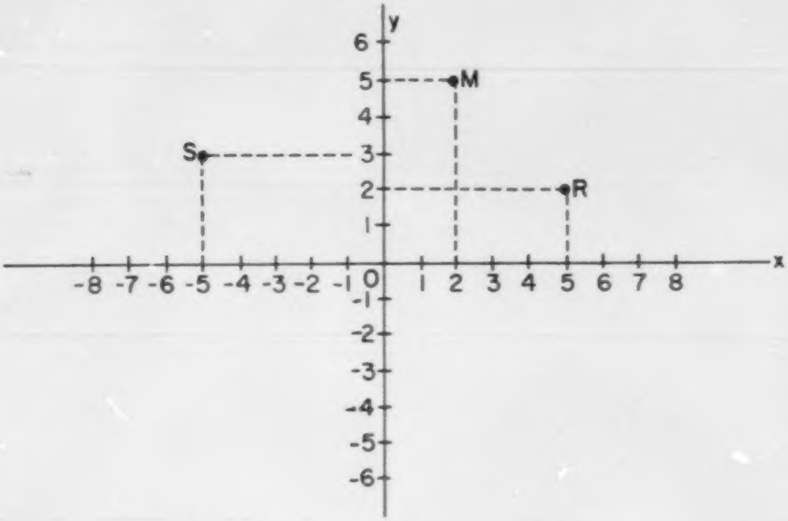


Figure 3

We see that the point R is located 5 units to the right of the y-axis and 2 units above the x-axis. If we start at the origin we can get to the point R by moving 5 units to the right and 2 units up. This tells that the coordinates of the point R may be written as the number pair

(5, 2).

In this case 5 is the abscissa and 2 is the ordinate. The point M is located 2 units to the right of the y-axis and 5 units above the x-axis. Once again if we start at the origin we can get to the point M by moving 2 units to the right and 5 units up. The coordinates of M are (2, 5). Do you see that the coordinates of S are (-5, 3)? Which of these two coordinates is the abscissa and which is the ordinate?

Check Your Reading

1. What are the coordinates of the point M in Figure 3?
2. What are the coordinates of the point R in Figure 3?
3. In the ordered pair (5, 2), where do you measure the 5? Where do you measure the 2?
4. Does the ordered pair of numbers (5, 2) name the same point as the ordered pair (2, 5)? Give reasons for your answer.
5. Which coordinate is also named the "abscissa"?
6. Which coordinate is also named the "ordinate"?

Problem Set 16-2b

1. Write the ordered pairs of numbers that are associated with points E, F, G, H, P, and Q in Figure 2, page 720.
2. Write the coordinates of the point which is:
   (a) to the right three units from the y-axis and up one unit from the x-axis.
   (b) to the left four units from the y-axis and down two units from the x-axis.

Problem Set 16-2b
(continued)

(c) to the right one unit from the y-axis and down six units from the x-axis.

(d) to the left three units from the y-axis and up five units from the x-axis.

3. In each of the following write the ordered pair which corresponds to the statement.
   (a) The ordinate is -2 and the abscissa is 5.
   (b) The y-coordinate is 3 and the x-coordinate is 0.
   (c) The abscissa is 3 and the ordinate is twice the abscissa.
   (d) The ordinate is three times the abscissa and the abscissa is -1.
   (e) The abscissa is 5 and the point lies on the x-axis.
   (f) The ordinate is -12 and the point lies on the y-axis.

The x- and y-axes separate the plane into four regions called quadrants. The quadrants are numbered I, II, III, and IV beginning with the right upper region and moving counter-clockwise as in Figure 4 below.

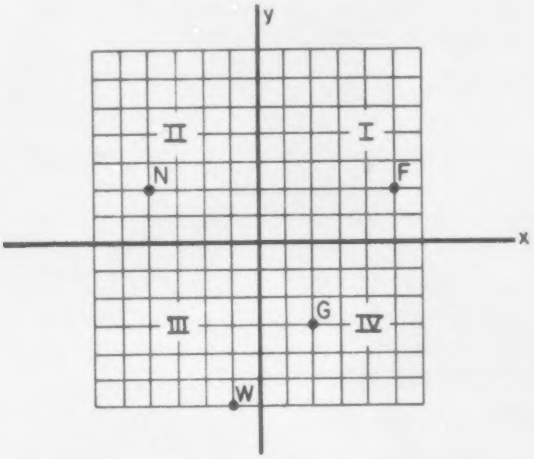


Figure 4

If a point is located in quadrant I, its coordinates are both positive numbers. In quadrant II, on the other hand, the x-coordinate, or abscissa, is a negative number, while the y-coordinate, or ordinate is a positive number. For example, the coordinates of the point F above are (5, 2) and the coordinates of the point N are (-4, 2).

What can you say about the coordinates of points in quadrants III and IV? For example, what are the coordinates of points W and G? Do you see that the coordinates of W are (-1, -6) and the coordinates of G are (2, -3)?

Points which lie on either of the axes are not in any of the quadrants. Examples of such points are those with coordinates (0, 5), (-2, 0), (0, 0), etc.

Check Your Reading

1. If a point is located in quadrant I, is its abscissa positive or negative?
2. In what quadrant does the point whose coordinates are (-4, 2) lie?
3. What are the coordinates of point W in Figure 4? In what quadrant does W lie?
4. In what quadrant does the point with coordinates (0, 5) lie?

Oral Exercises 16-2c

1. In Figure 4 is the x-coordinate of the point W positive or negative? of G?
2. In Figure 4 is the y-coordinate of the point W positive or negative? of G?
3. What are the coordinates of point W in Figure 4? of G?
4. Is the abscissa of any point in quadrant III positive or negative? In quadrant IV?
5. Is the ordinate of any point in quadrant III positive or negative? In quadrant IV?

Oral Exercises 16-2c
(continued)

6. In which quadrant might a point be if its x-coordinate is a negative number? Its y-coordinate a negative number?
7. In which quadrant (or quadrants) are both the abscissa and the ordinate positive numbers? Both negative numbers?
8. In which quadrant (or quadrants) is the x-coordinate a negative number and the y-coordinate a positive number?
9. In which quadrant (or quadrants) is the abscissa a positive number and the ordinate a negative number?

Problem Set 16-2c

From the following list of ordered pairs of numbers, sort the pairs and list them below following the name of the quadrant to which they belong:

Ordered pairs: (1, 4), (-2, 3), (4, -1), (-2, -1),
(-5, 7), (6, 2), (4, -9), (0, 0),
(3, -7), (7, 3), (-3, -7), (-7, -3)

1. All those in Quadrant I
2. All those in Quadrant II
3. All those in Quadrant III
4. All those in Quadrant IV.

16-3. The Graph of a Sentence in Two Variables.

We are now ready to draw the graph of the truth set of a sentence of the first degree in two variables. For convenience we shall, as before, call this the graph of the sentence.

Let us consider the sentence

$$2x - y + 1 = 0,$$

whose y-form is

$$y = 2x + 1.$$

This form enables us to describe the relation between x and y in the following words. "The ordinate is one more than twice the abscissa." The table below can be easily constructed from this description, or from the y form.

x	0	2	4	-2	-3
y	1	5	9	-3	-5

The plotted points would appear as in figure 5. You should check these carefully.

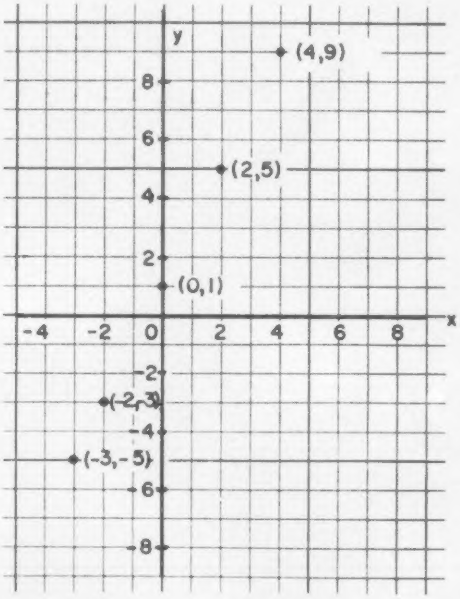


Figure 5

What do you notice about all of these points? Do they seem to form a definite pattern? Do they appear to lie on a straight line?

Suppose we draw a line connecting the point (4, 9) to the point (-3, -5) as in figure 6. Does this line pass through the other points?

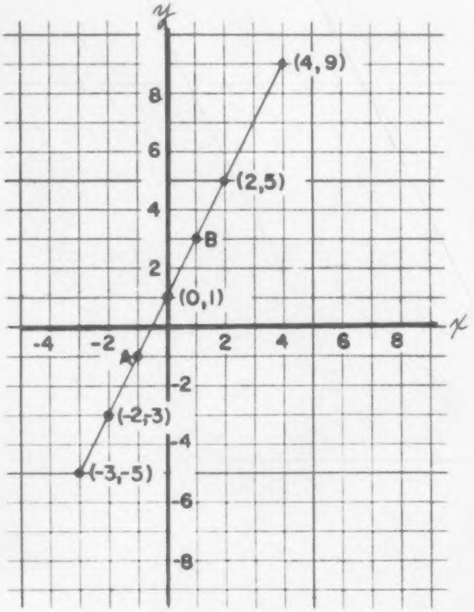


Figure 6

What does this suggest about any other point whose ordinate is 1 more than twice the abscissa? Would such a point lie on this same line provided that the line were extended in both directions? Suppose we consider the point with coordinates (3, 7). The ordinate is 1 more than twice the abscissa. Is this on the line?

Let us pick two more points on the line, the points A and B. Do you see that the coordinates of A are (-1, -1)? The coordinates of B are (1, 3). Both of these number pairs make

$$2x - y + 1 = 0$$

a true sentence.

Pick other points on the line. Do the coordinates of these points make the sentence true?

Thus, it appears that any point on the line has coordinates which make the sentence true. Although we could not possibly check all of them, it seems reasonable to assume that all points on our line have coordinates which satisfy the sentence

$$2x - y + 1 = 0.$$

We further assume that if our line were extended for any length in either direction, then the coordinates of all points on the extended line would also satisfy our sentence.

Let us consider another open sentence. In this example we shall also develop a convenient method for drawing graphs of sentences of this type. Take the sentence

$$x - y - 4 = 0.$$

We select any two number pairs which satisfy this sentence. We plot these points, and then draw the line connecting them. For example the number pairs

$$(5, 1) \qquad \text{and} \qquad (-2, -6)$$

make the sentence $x - y - 4 = 0$ true. Figure 7 shows the plotting of the points and the line through them.

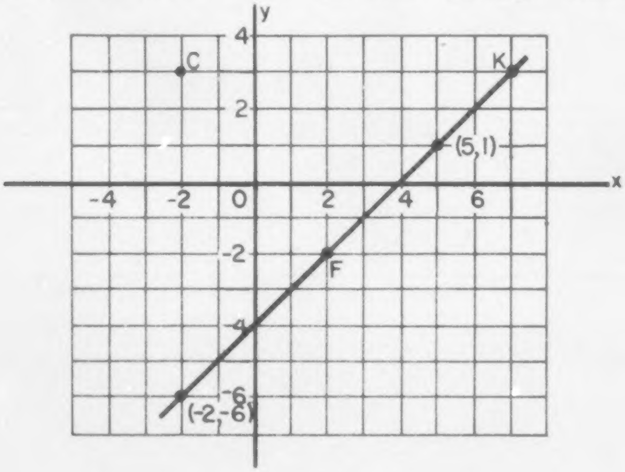


Figure 7

Points F and K are also on the line. The coordinates of F are (2, -2). The coordinates of K are (7, 3). Do the coordinates of these points satisfy the sentence $x - y - 4 = 0$?

Try other points. Of course you cannot try the coordinates of all points on the line. But what would you expect to be true if such a thing were possible?

We might look at this in another way. Suppose we start with any point on the line. Do you see that if we move one unit to the right and one unit up, we will then be on the line once more.

Now let's consider the sentence

$$x - y - 4 = 0 .$$

We know that the coordinates (5, 1) satisfy the sentence. What if we add 1 to each coordinate? This gives us (6, 2). Do the coordinates (6, 2) satisfy the sentence? What if we add 3 to each of the coordinates (5, 1)? Do the coordinates (8, 4) satisfy the sentence?

This suggests an idea. If we start anywhere on the line, move a certain distance to the right and then the same distance up, we shall end up on the line. On the other hand, if we start with coordinates which satisfy our sentence and add the same number to each coordinate the resulting coordinates still satisfy the sentence.

We thus find it very reasonable to conclude that all points on the line have coordinates which satisfy the sentence. It is also reasonable to assume that all number pairs which satisfy the sentence are coordinates of points on the line.

Now let's examine a point C which is not on the line. Its coordinates are (-2, 3). Does this number pair satisfy the sentence $x - y - 4 = 0$? The answer is no. Would this be the case for any point not on the line? Try some others!

What do these examples suggest? We have examined two sentences of the first degree in x and y, that is, sentences which could be written in the form

$$Ax + By + C = 0 ,$$

where A, B, and C are real numbers with A and B not both zero. The graphs of these sentences appeared to be lines. Number pairs which satisfied the sentences turned out to be coordinates of points on the lines. And points not on the lines had coordinates which did not satisfy the sentences. With this in mind we can readily accept the following statement.

> If an open sentence is of the form
>
> $$Ax + By + C = 0 ,$$
>
> where A, B, and C are real numbers with A and B not both zero, then its graph is a line.

We might also ask whether or not every line in a plane is the graph of a sentence of this type. The answer is yes. In fact it can be proved that for any two points in the plane there is one and only one sentence of the above form which the coordinates of these two points will satisfy.

As in the case of sentences in one variable, an open sentence in two variables acts as a sorter. It sorts the set of all ordered pairs of real numbers into two subsets: the set of ordered pairs which make the sentence true, and the set of ordered pairs which make the sentence false. In this sense we could say that an open sentence of the first degree in two variables "sorts" the points of a plane into two subsets: the points which lie on a particular line, and the points which do not.

Check Your Reading

1. Are the ordered number pairs (2, 5) and (4, 9) elements of the truth set of the open sentence $2x - y + 1 = 0$?
2. Is the point A whose coordinates are (-1,-1) on the line in Figure 6?
3. Do the coordinates of point A satisfy the equation $2x - y + 1 = 0$?

4. What are the coordinates of point B in Figure 6?

5. Do the coordinates of point B satisfy the equation $2x - y + 1 = 0$?

6. Is the line in Figure 6 the graph of the entire truth set of the sentence?

7. Are there any points that are not on the line in Figure 6 that satisfy the open sentence $2x - y + 1 = 0$?

8. In Figure 7, do the coordinates of the point F satisfy the open sentence $x - y = 4$? of K?

9. If an ordered pair of numbers makes an open sentence true, what can we say about the pair with respect to the graph of the truth set of the sentence?

10. If a point is on the graph of the truth set of a sentence, what can we say about the coordinates of the point with respect to the sentence?

11. What can we say about the coordinates of every point on a line that is the graph of the truth set of an open sentence in two variables?

Oral Exercises 16-3a

1. In each of the following complete the table of values so the resulting ordered number pairs will satisfy the given open sentence.

(a) $x - y = 2$

If x is	-2	6	2	0
then y is	-4			

(b) $x + y = -4$

If x is	0	1	3	-3
then y is	-4	-5		

(c) $x + y = 0$

If x is	2	0	4	-3
then y is	-2			

(d) $2x + y = 8$

If x is	1	2	0	-1
then y is	6			

(e) $x - 2y = 2$

If x is	4	0	2	-2
then y is	1			

Oral Exercises 16-3a
(continued)

2. Is the given ordered pair an element of the truth set of the sentence?

   (a) $(4, -1)$; $2x + y = 7$

   (b) $(0, -3)$; $3y - 2x = 6$

   (c) $(2, 3\frac{1}{2})$; $2y = x + 5$

   (d) $(4, -2)$: $3x = 8 - 2y$

   (e) $(9, 0)$: $x + 3y + 9 = 0$

   (f) $(3, 1)$; $2x + 4y - 10 = 0$

Problem Set 16-3a

1. (a) Plot 2 points whose coördinates satisfy the equation $x - y = 2$.

   (b) Draw the line through these two points.

   (c) Are the points that are represented by the coordinates $(2, 0)$ and $(0, -2)$ on the line that you drew in (b)?

   (d) Are there other points on this line?

   (e) Does the ordered number pair $(5, 3)$ satisfy the equation $x - y = 2$?

   (f) Plot the point that is represented by the ordered number pair $(5, 3)$. Is it on the line that you drew in (b)?

The two open sentences which we have studied so far have been

$$2x - y + 1 = 0$$

$$\text{and} \quad x - y - 4 = 0.$$

We have agreed that the graphs of these sentences are lines. It is not true that the graphs of all open sentences are lines. For example, if we were to consider the open sentence

$$y = x^2 + 1,$$

we would see that the following number pairs all satisfy the sentence, or equation.

$$(1, 2) \qquad (3, 10) \qquad (0, 1)$$

Check these to see that

$$2 = (1)^2 + 1$$
$$10 = (3)^2 + 1$$
$$1 = (0)^2 + 1$$

are all true. However, if you plot these points, you will see that they do not lie in a line. Try it.

There is a reason for this. It is due to the fact that this particular open sentence contains the term x^2 which, as you remember, we called a second degree term. On the other hand, note that equations such as

$$3x + 4y - 6 = 0$$

and

$$5y - x - 3 = 0$$

have first degree terms, but no terms whose degree is higher than the first. Such sentences, as we now know, have truth sets whose graphs are lines.

Check Your Reading

1. Is the graph of an open sentence in two variables in every case a line?
2. Is the graph of "$y = x^2 + 1$" a line?
3. Does the equation "$3x + 4y - 6 = 0$" contain any terms whose degree is higher than the first?

Problem Set 16-3b

1. For each of the following open sentences write three ordered number pairs which are elements of its truth set. Plot these points with respect to a pair of axes and draw the graph of the truth set of each sentence as in Figure 7. Remember, only two points are necessary to determine a line. The third point should be used as a check.

(a) $x + y = 6$
(b) $2x + y = 7$
(c) $x + 2y = 6$
(d) $x - y = 1$
(e) $2x - y = 2$
(f) $2x + 2y = 5$
(g) $x + 3y = 0$
(h) $x - y = 0$
(i) $2x - y - 3 = 0$
(j) $2x + 3y - 15 = 0$
(k) $2x + y = 2$
(l) $y = x + 1$
(m) $y - 2 = x$
(n) $y = -3x$

2. (a) Write three ordered number pairs which satisfy the sentence $2x + y = 4$.
   (b) Draw the graph of the sentence.
   (c) Write two ordered number pairs that do not satisfy the sentence.
   (d) Plot the points corresponding to the number pairs that were written in (c) with respect to the same set of axes on which the graph for 2(a) was plotted.
   (e) Are these two points on the graph of the sentence?

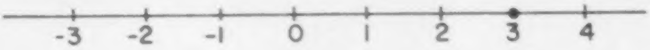
3. (a) Find five ordered number pairs which satisfy the sentence $2x^2 - y = 0$.
   (b) Plot the points found in (a). Do they lie on a line?
   (c) Does the sentence in (a) contain any terms whose degree is higher than the first?

4. (a) Find five ordered number pairs which satisfy the equation $x - y^2 = 0$.
   (b) Plot the points found in (a). Do they lie on a line?
   (c) Does the equation in (a) contain any terms whose degree is higher than the first?

When we were asked in Chapter 3 to draw the graph of the sentence

$$x = 3$$

on the number line, we found that its graph was a single point.



However, suppose we are asked to consider this same sentence as a sentence in two variables and draw its graph on the number plane. This might be puzzling at first, since the sentences whose graphs we have been plotting on the number plane have all contained the variable y as well as the variable x; also the elements in our truth set have all been number pairs rather than individual numbers.

Our problem can be solved if we consider the sentence

$$x = 3$$

as though it had been written

$$x + 0y - 3 = 0.$$

You recall that in writing the general form of a first degree sentence in x and y as

$$Ax + By + C = 0$$

it was stated that A, B, and C were real numbers, but that A and B were not both zero. In this sense we do have an open sentence in two variables, even though the coefficient of the variable y is zero. We can now consider the set of number pairs which will make the new sentence true. What number pairs will satisfy the sentence

$$x + 0y - 3 = 0\ ?$$

Since we are working in the number plane, we must think in terms of two variables even though the y variable did not actually appear in the original form "$x = 3$". It should be easy to see that the following number pairs satisfy our sentence:

(3, 1) (3, 3) (3, 4) (3, 5),

since the sentences

$$3 + (0)(1) = 3$$

$$3 + (0)(2) = 3$$

$$3 + (0)(3) = 3$$

are all true by the multiplication and addition property of zero.

Do you also see that any pair of the form

$$(3, n),$$

where n represents any specified number, will be a member of our truth set?

Our graph is easy to draw. As before, we select two points, say $(3, 1)$ and $(3, 5)$. Plot these points. Then draw the line connecting them. Figure 8 illustrates the graph.

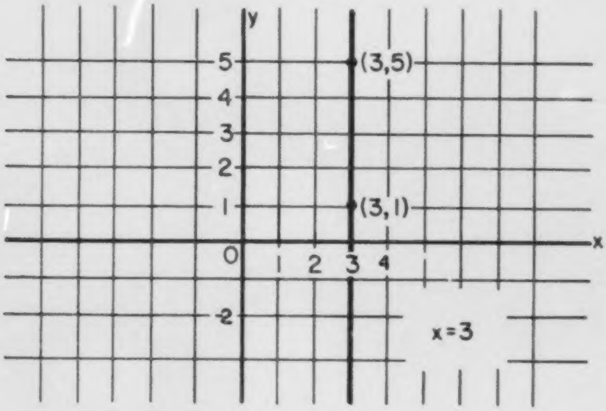


Figure 8

In a similar way let us now consider the sentence

$$y = 5.$$

Do you see that this can be thought of as

$$0x + y - 5 = 0, \quad \text{or} \quad y = 0x + 5,$$

if we wish to draw its graph on the number plane? Some number pairs in the truth set are

$$(1, 5) \qquad (-6, 5) \qquad (4, 5).$$

The graph is given in Figure 9.

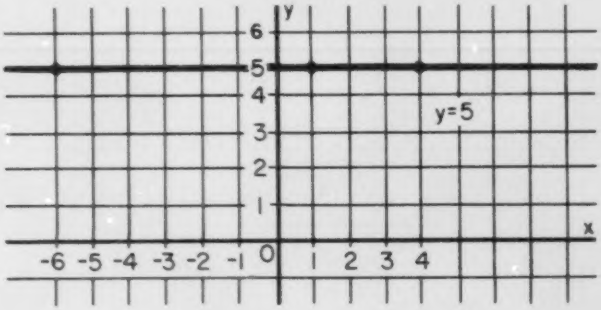


Figure 9

Figures 10 and 11 illustrate graphs of sentences of this same type. You should check to see that these graphs are properly drawn.

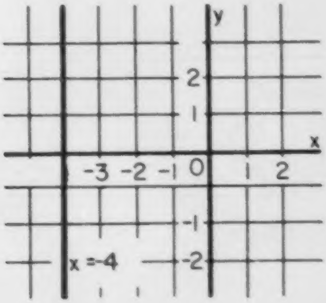


Figure 10

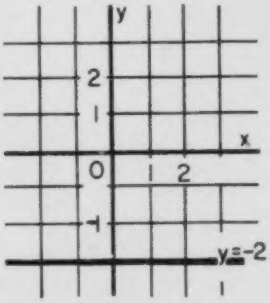


Figure 11

Check Your Reading

1. What is another way of writing the sentence $x = 3$ as a sentence in 2 variables?
2. Number pairs of what form will satisfy the sentence $x + 0y - 3 = 0$?
3. Do the ordered number pairs (3, 1), (3, 2), (3, 3), (3, 4), and (3, 5) satisfy the sentence $x + 0y - 3 = 0$?
4. What is another way of writing the sentence $y = 5$ as a sentence in 2 variables?
5. Name three number pairs that will make the open sentence $0x + y = 5$ true.

Oral Exercises 16-3c

1. Describe in your own words the graphs of the following sentences.

(a) $x = 3$
(b) $x = -4$
(c) $y = 5$
(d) $y = -2$
(e) $x + 3 = 0$
(f) $x + (-7) = 0$
(g) $y + (-3) = 0$
(h) $y + 3 = 0$
(i) $x = 0$
(j) $y = 0$

Problem Set 16-3c

1. Draw the graph of the truth set of each of the following open sentences.

(a) $x = 5$
(b) $x = -5$
(c) $y = 2$
(d) $y = -2$
(e) $x = 1$
(f) $x = -3$
(g) $y = -8$
(h) $y = 8$
(i) $x = 0$
(j) $y = 0$

2. With respect to the same set of axes draw the graphs of the following equations.

$$y = 3x + 5$$
$$y = 2x + 5$$

Problem Set 16-3c
(continued)

$$y = x + 5$$
$$y = 5$$

3. With respect to the same set of axes draw the graphs of the following equations:

$$y + 4x = 0$$
$$x + y = 0$$
$$y = 0$$
$$3x + 2y = 0$$

What do you notice about the graphs? What is special about the equations?

4. With respect to the same set of axes draw the graphs of the following sentences:

$$y = 2$$
$$y = 2x + 2$$
$$y = 3x + 2$$
$$y = \frac{1}{3}x + 2$$

What do you notice about the graphs? What is common to all the equations?

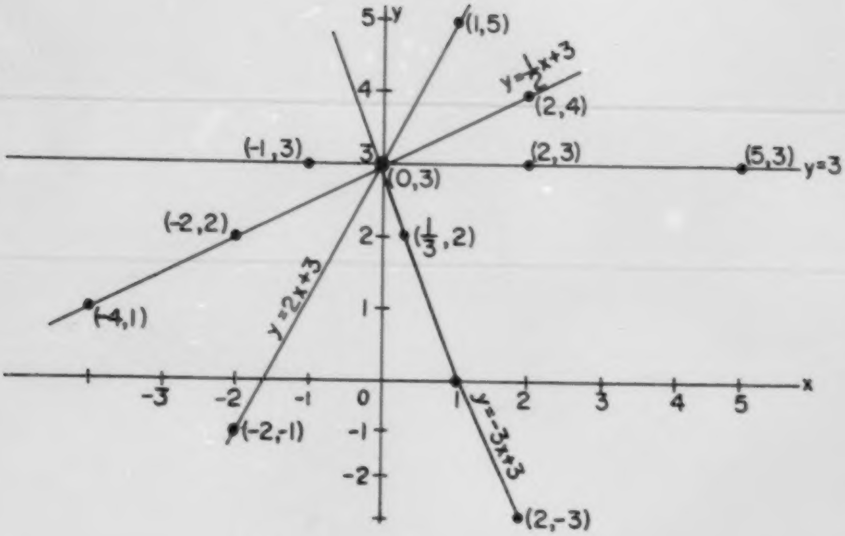
16-4. Intercepts and Slopes.

In the last group of exercises you were asked to notice something special about the graphs. We shall give a further example to emphasize this idea.

Consider the sentences:

$$y = 3$$
$$y = 2x + 3$$
$$y = \frac{1}{2}x + 3$$
$$y = -3x + 3$$

Notice that these sentences are all in y-form and that they have something else in common. What is it? Let us examine the graphs of these sentences.



The most striking feature of the graphs is that they all meet at the same point, (0, 3). This point is on the y-axis and is called the y-intercept of each line. Now look at the y-form of each sentence. Do you see that in each case the constant term on the right side is 3? It should be clear that this number determines the y-intercept of the line, that is, the point at which the line intersects the y-axis. We saw that the coordinates of this point are (0, 3). Thus we can say that if a sentence is in y-form,

$$y = ax + b,$$

then (0, b) is the y-intercept of the graph of the sentence; that is, the graph includes the point whose coordinates are (0, b). For example, for the sentence $y = 5x - 3$, the y-intercept of the graph is the point (0, -3).

Check Your Reading

1. The sentences "$y = 3$," "$y = 2x + 3$," and "$y = 3x + 3$" are in y-form. What else do they have in common?
2. The graph of the sentence "$y = \frac{1}{2}x + 3$" contains the point which has coordinates (0, 3). What is this point called?
3. What is the y-intercept of the line whose equation is $y = 5x - 3$?

Oral Exercises 16-4a

State the y-intercept of the graph of each of the following sentences. State the y-form of the sentence first if necessary.

1. $y = 3x + 2$
2. $y = 2x - 5$
3. $y = \frac{1}{2}x + 12$
4. $y = 7x - \frac{3}{4}$
5. $x + y = 2$
6. $2x + y = 7$
7. $\frac{3}{4}x + y = \frac{9}{10}$
8. $.8x + y = .16$
9. $x - y = 12$
10. $2x - y = \frac{1}{3}$
11. $2x + 2y = 5$
12. $x + 3y = 6$
13. $x - 2y = 4$
14. $3x + 4y = 8$
15. $2x + 3y = 5$

Problem Set 16-4a

Write each of the following sentences in y-form; then draw the graph of the sentence using the y-intercept as one of the points of the graph.

1. $2x + y = 5$
2. $3x - y = 6$
3. $4x + 2y = 8$
4. $3x + 2y = 9$
5. $7x - 2y = 4$
6. $4x - 3y = 5$

Draw the graphs of the following sentences with respect to the same set of axes as above.
Look at the y-form and look at the graph. What do these graphs seem to have in common? What do the y-forms have in common?

Problem Set 16-4a
(continued)

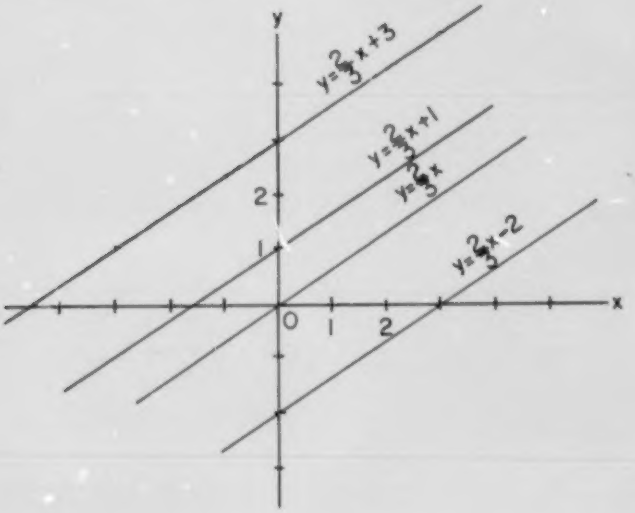
7. $-2x + y = 5$
8. $-6x + 3y = 1$
9. $10x - 5y = 3$
10. $-8x + 4y = -1$

If you did the last four exercises carefully you should have noticed another interesting relationship between the y-form of a sentence in two variables, and the graph of the sentence.

Let us emphasize this relationship by considering another group of sentences:

$$y = \frac{2}{3}x$$
$$y = \frac{2}{3}x + 1$$
$$y = \frac{2}{3}x - 2$$
$$y = \frac{2}{3}x + 3$$

What is "special" about the y forms of these sentences? Do you see that the coefficients of x are all the same, namely $\frac{2}{3}$?



What do you notice that is "special" about these graphs? Do you see that the lines are parallel? It appears then that sentences in the y-form which have the same coefficient of x have graphs which are parallel to each other.

Check Your Reading

1. The sentences "$y = \frac{2}{3}x$", "$y = \frac{2}{3}x + 1$", and "$y = \frac{2}{3}x - 2$" are in y-form. What else do they have in common?
2. What is special about the graphs of sentences in y-form which have the same coefficient of x?

Oral Exercises 16-4b

In each of the following problems, three equations in two variables are given.

(a) Give the y-intercept of each line.

(b) Tell which equations have graphs which are parallel to each other.

(c) Tell which equations have graphs not parallel to any others.

(d) Tell which equations have the same graph.

1. $y = 3x + 2$
   $y = 3x - 5$
   $y = 2x + 2$

2. $y = \frac{1}{2}x - 2$
   $y = \frac{1}{2}x + 2$
   $2y = x - 4$

3. $y = -3x + 7$
   $y = -3x - 2$
   $-\frac{1}{3}y = x + \frac{2}{3}$

4. $x + y = \frac{3}{4}$
   $x - y = \frac{3}{4}$
   $3x - y = \frac{3}{4}$

Problem Set 16-4b

In each of the following cases draw the graph of the first equation given; then select from the list following it only those equations that have graphs parallel to the first and draw their graphs.

1. $y = 3x + 2$
   $y = 3x + 5$
   $y = 3x - 1$
   $y = 4x + 2$

2. $y = \frac{1}{2}x - 5$
   $2y - x = -5$
   $y = \frac{1}{2}x + 2$
   $x - 2y = 4$

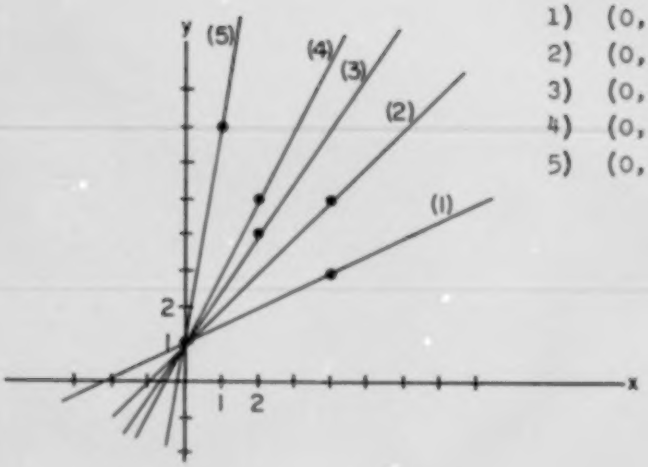
3. $y = -2x + 1$
   $y = 3x - 2$
   $y + 2x = 0$
   $y = x$

4. $y = x + 3$
   $y + x = 3$
   $y - x = 2$
   $y = x + 5$

Let us now examine another group of sentences. These are similar to the ones we studied in connection with the y-intercept.

1) $y = \frac{1}{2}x + 1$
2) $y = x + 1$
3) $y = \frac{3}{2}x + 1$
4) $y = 2x + 1$
5) $y = 6x + 1$

For each of these sentences we will draw the graph by means of two points. As a convenience we use the y-intercept in each case as one of our points. The second point also appears at the right of the list on the following page.



1) (0, 1) (4, 3)
2) (0, 1) (4, 5)
3) (0, 1) (2, 4)
4) (0, 1) (2, 5)
5) (0, 1) (1, 7)

Each line is labeled by number. First, note the equations. We see that in every case they are the same except for the coefficient of x. We also see that the coefficient of x increases with each new sentence. Looking at the graphs we see that each line seems to get progressively steeper. Another way of saying this might be to state that in each case the "slope" of each line appears to increase.

It seems natural, therefore, to make the following statement with reference to the idea of slope.

> The slope of a line is the coefficient of x in the corresponding sentence written in the y-form.

When we say that the slope of a line is very steep, or not so steep, we are speaking vaguely. It is now possible to describe the slope of a line in a very definite way, using in each case a specific number.

For example, if we are given a line whose corresponding equation is

$$y = \frac{2}{3}x + 5,$$

we can say very definitely that the slope of this line is

$$\frac{2}{3}.$$

To sum this up we can now say that for a sentence of the form

$$y = ax + b$$

where a and b are specified real numbers, the slope of the corresponding line is the number a and the y-intercept is the point $(0, b)$.

Thus the y-form reveals two specific bits of information about the graph of the sentence. It gives us the slope and the y-intercept.

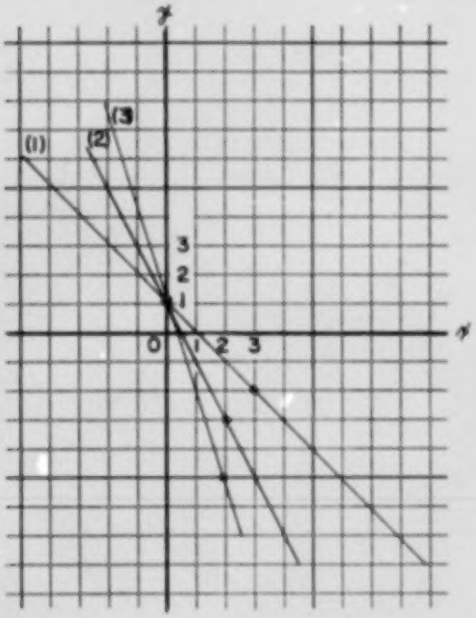
In the sentences

1) $y = -x + 1$,

2) $y = -2x + 1$,

and 3) $y = -3x + 1$

the coefficients of x in the y-form are negative. Here we think of $-x$ as $(-1)x$. Thus by definition the slopes of the corresponding lines are negative. The graphs appear below. As in the previous drawing the lines are numbered. Each has been determined by the indicated pair of points.



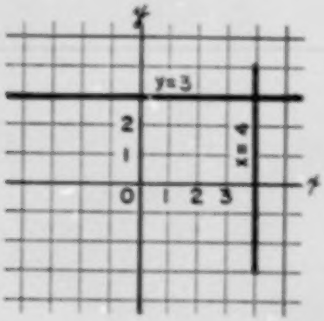
1)	(0, 1)	(3, -2)
2)	(0, 1)	(2, -3)
3)	(0, 1)	(2, -5)

With respect to the idea of steepness we note the following. The steepest line appears to be the line whose slope has the greatest absolute value. Do you see that this is true for lines of positive slope also?

In a previous section we have studied lines of the form

$$y = 3 \qquad \text{and} \qquad x = 4,$$

the graphs of which are shown below.



We noted that the sentence $y = 3$ could be written as

$$0x + y - 3 = 0,$$

with y-form $$y = 0x + 3.$$

According to our definition, what is the slope of the line which is the graph of this sentence? Do you see that it is zero? This is reasonable in terms of "steepness". Since the line is horizontal it has, in a sense, zero steepness.

Sentences of the type

$$x = 4,$$

however, present a special problem with respect to slope, since this type of sentence can not be written in y-form. Do you see why? We do not define slope for lines corresponding to such sentences. We have already noted that lines of this type are vertical.

Check Your Reading

1. Compare the graphs of "$y = \frac{1}{2}x + 1$" and "$y = \frac{3}{2}x + 1$". Which has the y-intercept $(0, 1)$? Which is steeper?
2. How is the slope of a line determined from the corresponding equation in y-form?
3. What is the slope of the line whose equation is

   "$y = \frac{2}{3}x + 5$"?
4. Which has the steeper line as its graph, "$y = -3x + 1$" or "$y = -x + 1$"?
5. What form of "$y = 3$" shows the slope of its graph?
6. What lines have zero slope?
7. Why is it impossible to write a y-form for "$x + 0y - 4 = 0$"?

Oral Exercises 16-4c

1. For each of the following equations, give the slope and the y-intercept of the graph.

   (a) $y = \frac{3}{4}x + 1$
   (b) $y = -3x + 3$
   (c) $y = 17x - 6$
   (d) $y = -\frac{2}{3}x$
   (e) $y = 5$
   (f) $x = -2$

2. Which of the following equations have graphs which are parallel to each other?

   (a) $y = 2x - 1$
   (b) $y = \frac{1}{3}x + 8$
   (c) $3y = x$
   (d) $y = 2x$
   (e) $y - 2x = 5$
   (f) $y = 3x + 5$

Problem Set 16-4c

1. In each of the following give the slope and the y-intercept of the line whose equation is given.

   (a) $y = 2x + 1$
   (b) $2x - y + 1 = 0$
   (c) $\frac{x}{2} + \frac{y}{3} = 1$
   (d) $x = -3y + 1$
   (e) $2x + 3y = 6$
   (f) $x = 5$

Problem Set 16-4c
(continued)

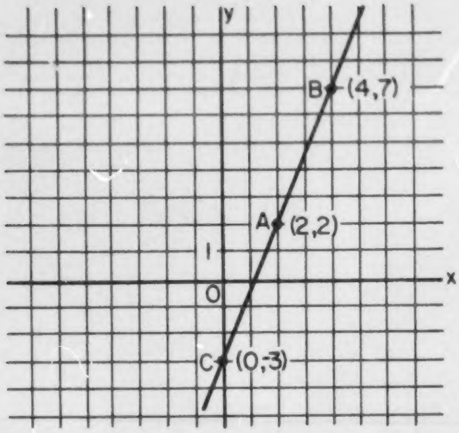
2. Write the equation of each line, given the following information, and draw its graph.
   - (a) the slope is 2 and the y-intercept is $(0, 6)$
   - (b) the slope is $-\frac{1}{2}$ and the y-intercept is $(0, 0)$
   - (c) the slope is 6 and the y-intercept is $(0, -1)$
   - (d) the slope is $\frac{1}{12}$ and the y-intercept is $(0, -2)$
   - (e) the slope is 0 and the y-intercept is $(0, -3)$
   - (f) the line is vertical and passes through the point $(3, 5)$.
3. Write the equation of a line which is parallel to "$y = 3x + 8$" and which has the y-intercept $(0, 1)$.
4. Write the equation of a line which is parallel to "$y = -2x - 3$" and which has the y-intercept $(0, 3)$.
5. If a_1 and a_2 are the slopes of parallel lines then what is the relationship between a_1 and a_2?

We have already observed that the word slope as it is commonly used suggests a measure of the steepness of a line. It gives an idea of the rate at which a line gains or loses in height as we move along it in a certain direction. We will see that slope by our definition is closely related to this type of measurement also.

Consider the following line. Its equation is

$$y = \frac{5}{2}x - 3.$$

The graph is shown on the following page.



The coordinates of point A are (2, 2). The coordinates of point B are (4, 7). Let us now check the horizontal distance from A to B, and the vertical distance from A to B. We see that the horizontal distance is 2. This can be found by subtracting the abscissa of A from the abscissa of B, that is

$$4 - 2 = 2.$$

The vertical distance, which we can obtain by subtracting ordinates, is 5, that is

$$7 - 2 = 5.$$

Let us now form the ratio

$$\frac{\text{Vertical Distance from } A \text{ to } B}{\text{Horizontal Distance from } A \text{ to } B}.$$

Here the word ratio has the same meaning as fraction. It indicates the quotient of two numbers. We see that the value of this ratio is

$$\frac{5}{2}.$$

Let us now consider the two points C and B. If we subtract the abscissa of C from the abscissa of B to find the horizontal distance from C to B, we obtain

$$4 - 0 = 4.$$

For the vertical distance we get

$$7 - (-3) = 10.$$

Again we form the ratio

$$\frac{\text{Vertical Distance from } C \text{ to } B}{\text{Horizontal Distance from } C \text{ to } B}.$$

This is equal to $\frac{10}{4}$, another name for

$$\frac{5}{2}.$$

If we now consider the points C and A and form the same type of ratio, we will find that

$$\frac{\text{Vertical Distance from } C \text{ to } A}{\text{Horizontal Distance from } C \text{ to } A} = \frac{5}{2}.$$

Check this!

Two ideas are suggested by all of this. Do you see what they are? The value of the ratio in all three cases is the same. It is $\frac{5}{2}$. This is also the coefficient of x in the y-form of the equation, in other words, the slope.

Another point on the same line is $(8, 17)$. Call this point D. Now consider the two points A and D. Form the ratio of the vertical distance divided by the horizontal distance as before. This becomes

$$\frac{17 - 2}{8 - 2}.$$

Is this another name for $\frac{5}{2}$?

Let us consider one more equation.

$$y = -2x + 3.$$

Two points on the line selected at random are $(-5, 13)$ and $(2, -1)$. Do you see that these are elements in the truth set? If we subtract the ordinate of the second point from the ordinate of the first point we get 14. Doing the same thing with the abscissas in the same order we get -7. The ratio is

$$\frac{14}{-7}$$

which is the same as

$$-2.$$

What is the coefficient of x in our equation?

The above examples illustrate a relationship which can be stated in a general way as follows:

> If (c, d) and (e, f) are any two points on a line with $c \neq e$ and if the y-form of the corresponding equation is
>
> $$y = ax + b,$$
>
> then the ratio $\frac{d - f}{c - e}$ is equal to the real number a, that is, the ratio is equal to the slope of the line.

This statement can be proved to be true for all lines whose equations can be put into the y-form. Since the proof involves mathematical ideas which we have not yet studied we shall not develop the proof at this time.

A word should be said about the order in which the two points are selected. Suppose we consider the points $(-5, 13)$ and $(2, -1)$ of the previous example. In forming the ratio we subtracted the coordinates of the second point from the coordinates of the first. If we subtract the first coordinates from the second, we obtain the ratio

$$\frac{-1 - 13}{2 - (-5)}$$

which is equal to

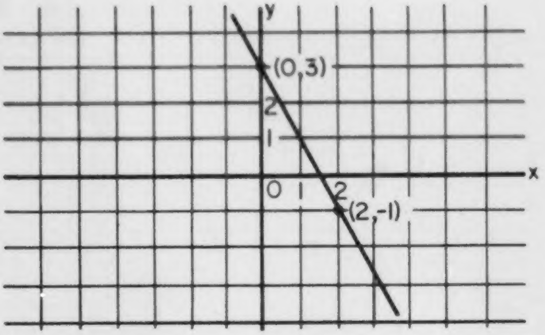
$$\frac{-14}{7}.$$

But this is also another name for

$$-2.$$

It is interesting to investigate further the meaning of a negative number as the slope of a line. In the graph of our equation

$$y = -2x + 3,$$



we see that the line appears to slant downward as we scan it from left to right. In other words, as we assign increasing values to x, the corresponding values of y decrease. In this case the slope is a negative number.

In the case of an equation such as

$$y = 3x + 5$$

we observe that as we assign increasing values to x, the corresponding values of y increase. We see that in this case the slope is a positive number.

Check Your Reading

1. If A and B are two points on the graph of "$y = \frac{5}{2}x - 3$", then what is the ratio of "the vertical distance from A to B" to "the horizontal distance from A to B"?
2. The points $(2, 2)$ and $(8, 17)$ both lie on the graph of "$y = \frac{5}{2}x - 3$". How is the vertical distance from the first point to the second point found? How is the horizontal distance from the first point to the second point found?
3. What is the slope of a line which contains the points $(-5, 13)$ and $(2, -1)$?
4. Complete the sentence: A line which has a negative slope slants __________ as we scan it from left to right.

Oral Exercises 16-4d

1. Find the slope of the line through each of the following pairs of points.

   (a) (2, 3) and (3, 6)
   (b) (6, 2) and (3, -1)
   (c) (1, 1) and (-2, -2)
   (d) (-7, -3) and (6, 2)
   (e) (-7, 3) and (8, 3)
   (f) (8, 6) and (-4, -1)

2. Indicate in which of the following cases the values of y increase as we assign increasing values to x. In which cases do the values of y decrease as we increase the values of x?

   (a) $y = 2x$
   (b) $y = -3x$
   (c) $y = x + 1$
   (d) $y = -\frac{1}{3}x + 2$
   (e) $x + y = 1$
   (f) $2x - y = 6$

Problem Set 16-4d

1. Find the slope of the line through each of the following pairs of points.

   (a) (3, -12) and (-8, 10)
   (b) (4, 11) and (-1, -2)
   (c) (6, 5) and (2, 0)
   (d) (0, 0) and (-6, -2)
   (e) (3, 5) and (6, 5)
   (f) (-3, 4) and (-3, 6)

2. Draw the following lines.

   (a) Through the point (-1, 5) with slope $\frac{1}{2}$
   (b) Through the point (2, 1) with slope $-\frac{1}{2}$
   (c) Through the point (3, 4) with slope 0
   (d) Through the point (-3, 4) with slope 2
   (e) Through the point (-3, 4) and parallel to the y-axis

3. Write the equation of the line, given the following information:

   (a) slope 3, y-intercept (0, 3)
   (b) point (1, 6), y-intercept (0, 3) (first find the slope)
   (c) point (-1, 1), y-intercept (0, 3)

Problem Set 16-4d
(continued)

(d) point (2, 6), containing the origin (0, 0)
(e) slope -2, y-intercept $(0, \frac{4}{3})$
(f) point (-2, 4), y-intercept (0, 1)

16-5. Graphs of Sentences Involving an Order Relation.

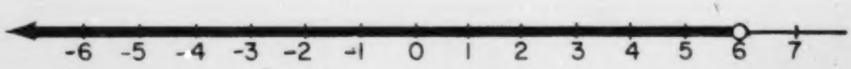
In Chapter 3 we studied open sentences in one variable involving an order relation, such as

$$x + 3 < 9.$$

The equivalent sentence

$$x < 6$$

has the following graph:



We now ask the questions:

(1) "How is the truth set determined if the sentence of the first degree involving an order relation contains two variables?"

(2) "How do we construct the graph of such a sentence?"

Let's begin with an example such as

$$x + y > 3 .$$

It will help us if we first draw the graph of the equation

$$x + y = 3 ,$$

which, as you see, is formed by merely replacing the symbol ">" of our inequality by an "equals" sign. Do you see that the graph of this equation is the line which appears in Figure 12?

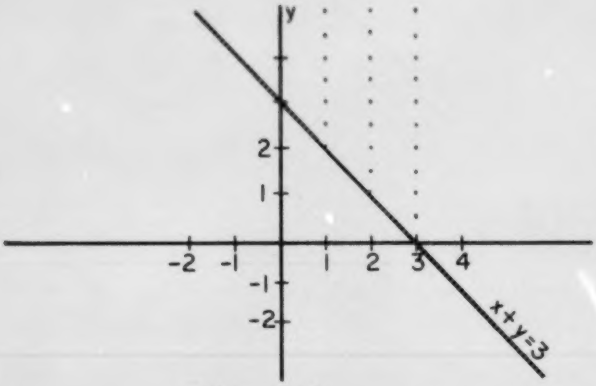


Figure 12

As we know, all number pairs which satisfy our equation have their corresponding points on this line. We also know that if a point is not on this line, then its coordinates will not satisfy the equation.

Furthermore, it should be clear that the line separates the points of the plane into three sets. These three sets are 1) the points lying on one side of the line, 2) the points lying on the line itself, and 3) the points lying on the other side of the line.

Let us now return to our original sentence, the inequality

$$x + y > 3.$$

We wish to determine its graph. We first note that if a number pair satisfies the equation

$$x + y = 3$$

then it cannot make the inequality true. Can you name the property of order which tells us this? Thus we see that no point of our truth set can lie on the line, that is, our points must lie on one side or the other.

Suppose now we let x be a fixed number; for example, let $x = 1$. We want to see what values of y when paired with the number 1 will make our sentence true. It is easy to determine this if we substitute 1 for x in the sentence. This gives us

$$1 + y > 3 .$$

Do you see that an equivalent sentence is

$$y > 2 ?$$

From this we can see that if y is any number greater than 2, then the number pair

$$(1, y)$$

will be in the truth set of the sentence

$$x + y > 3 .$$

In the same way we can show that if $x = 2$, then if y is any number greater than 1, the number pair

$$(2, y)$$

will be in our truth set.

Likewise if $x = 3$, then any number y which is greater than zero will make the pair

$$(3, y)$$

an element in the truth set.

Our graph shows many of these points. You will note that the points all lie on one side of the line. In this particular case they appear to be above the line. It can be shown that no points on the other side of the line have coordinates which satisfy the sentence.

It should not, at this time, be difficult to form an idea about the graph of the sentence we have been working with. Can you suggest a way of describing the correct graph?

It is the set of all points which lie on the same side of the line as those we have already indicated. To show this we can shade this side. Remember the line is not part of the graph. This can be demonstrated by making it a "broken" line, as in Figure 13.

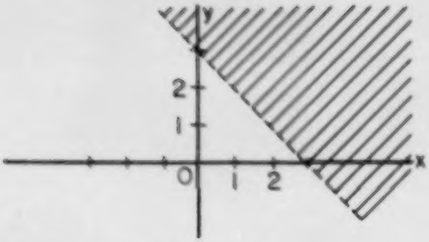


Figure 13

Example. Draw the graph of the sentence

$$2x - 3y > 10 .$$

As before, we begin by constructing the graph of the equation

$$2x - 3y = 10 .$$

This gives us the following line.

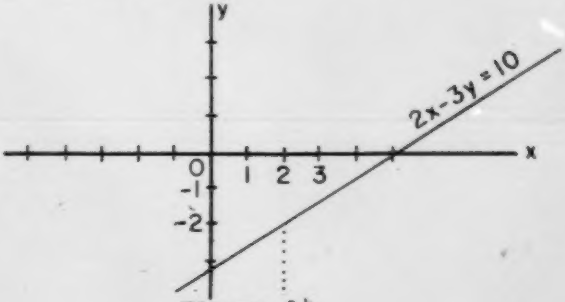


Figure 14

We now return to the inequality and let x assume a fixed value. For convenience we can choose the number 2. Our inequality can then be written as

$$4 - 3y > 10 .$$

In order to determine a set of corresponding truth values for y we must find an equivalent inequality. As we learned in Chapter 15, this can be done as follows:

We see that an equivalent sentence is

$$- 3y > 6 \text{ , (Why?)}$$

and this becomes $-2 > y$, (Why?)

which can be written as $y < -2.$

This tells us that any point with an abscissa of 2 and an ordinate which is less than -2 is a point in our graph. Some of these points are shown in Figure 14. Without having to locate any more specific points we can shade in the side of the plane in which these points are found, as in Figure 15.

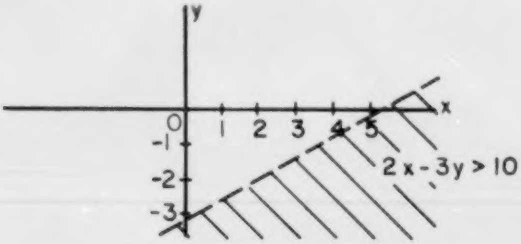


Figure 15

The method we have been using to construct graphs of inequalities of the first degree in two variables can be summarized as follows: We first determine the graph of an associated equation, that is the equation obtained by replacing the inequality symbol with an "=". We then let x assume a convenient value and substitute this in the original inequality. This will give us an inequality in the one variable, y. We obtain the truth set of this sentence. Its graph will lie on one side of the line determined by the associated equation. The graph of our original sentence is the set of all points in the plane located on this same side of the line.

In our two examples the line of the equation was used to locate our points, but it was not part of the graph. However, if our sentences had contained the symbol "$\geq$" meaning "greater than or equal to", that is, if they had been written

$$x + y \geq 3$$

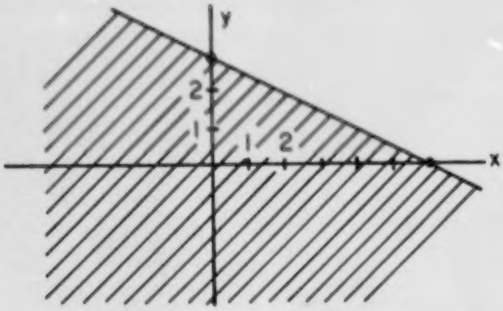
and

$$2x - 3y \geq 10 ,$$

then the graphs would have included the lines. In such cases the lines in the drawings would not have been broken. For example, the sentence

$$x + 2y \leq 6$$

has the graph:



Check Your Reading

1. Into how many sets does the line "$x + y = 3$" separate the plane?
2. What property of order tells us that if a number pair satisfies the equation "$x + y = 3$" then it cannot satisfy "$x + y > 3$"?
3. How many numbers satisfy "$1 + y > 3$"?
4. What is the meaning of a broken line in the drawing of the graph of "$x + y > 3$"?

Problem Set 16-5

1. In each of the following draw the graph of the sentence by first drawing the graph of an associated equation.
   - (a) $x - y > -5$ (the associated equation is $x - y = -5$)
   - (b) $2x - y < 6$
   - (c) $3x - y \geq 0$
   - (d) $3x - 2y \leq 12$
   - (e) $y < -2$
   - (f) $x \geq 4$

Problem Set 16-5
(continued)

2. Draw the graphs of the following on the same axes.

   (a) $y = x + 2$

   (b) $y > x + 2$

   (c) $y < x + 2$

3. Do the graphs of Problem 2 use every point in the plane? What is the equation of the line which separates the plane into 3 sets?

Summary

In this chapter we have studied the truth sets of open sentences in two variables and the graphs of these sentences. We learned that each element of the truth set of a sentence of the form

$$Ax + By + C = 0$$

is an ordered number pair, the first number representing a value of x and the second a value of y.

We have learned that a number pair such as $(2, 5)$ can be associated with a particular point on a plane, and that this point can be plotted by means of two perpendicular number lines called axes. We saw that to every ordered pair of numbers there corresponds a point in the plane and to every point there corresponds a pair of numbers. Furthermore it was shown that all points which satisfy an equation such as the one above lie on a line. This fact enables us to construct the graph of such an equation. We also learned that every line in the plane is the graph of a first degree equation in two variables, and that two points determine this equation.

We said that if a sentence is written in the y-form

$$y = ax + b,$$

then a is the slope of the graph of the sentence and $(0, b)$ is the y-intercept of the graph of the sentence.

If A and B are two points on the graph of the sentence then the slope of the graph is equal to the ratio:

$\frac{\text{vertical distance from A to B}}{\text{horizontal distance from A to B}}$. The y-intercept is the point on the y-axis where the graph of the sentence intersects the y-axis.

Finally, the truth sets of sentences of first degree in two variables involving inequalities have been shown to consist of the coordinates of all points lying on one side of a specific line. This line is determined by replacing the "$<$" or "$>$" symbols with an equality "$=$". When the order relation is "$\geq$" or "$\leq$" the line must be included in the graph.

Review Problem Set

1. Simplify each of the following:

(a) $(-2) + (-6)$

(b) $-7 - 4$

(c) $-5 + 8 - 14 + 22$

(d) $\frac{2}{3} + \frac{1}{4} + \frac{5}{6} - \frac{10}{8}$

(e) $16a - 5b + 14a - 10a + 5b$

(f) $3a + 6b - 5a + c$

(g) $\dfrac{\frac{3}{5}}{\frac{1}{10}}$

(h) $\dfrac{-\frac{4a}{6c^2}}{\frac{3m}{2c}}$, where $c \neq 0$ and $m \neq 0$

(i) $\frac{2am^2}{9yz^2} \cdot \frac{24ay^4z^5}{14m^5z}$, where $m \neq 0$, $y \neq 0$, and $z \neq 0$

(j) $(a + 3b) - (2a + 6b)$

(k) The result of subtracting $m - 2n + 4$ from $3m - 6$

(l) $5x + 2y - 3z(3x - 3y - 6z)$

Review Problem Set
(continued)

2. Factor each of the following polynomials into prime factors.

(a) $am + an$
(b) $6m + 6n$
(c) $14a^2 + 42a$
(d) $18c + 12cd$
(e) $9 - 6a$
(f) $6a^2b + ab^2$
(g) $3m - 2n$
(h) $3x - 9$
(i) $5m + 6m + 9m$
(j) $a + a^2 + a^3$
(k) $by^2 + by - b$
(l) $2\pi r - 2\pi t$
(m) $20y - 15x - 10$
(n) $4y + 5x + 7z$
(o) $2ac - 4ab - 2a$
(p) $ax + x + ay + y$
(q) $cy - cx + ay + ax$
(r) $mn^2 + n + m^2n + m$
(s) $-8bc + 6ac + 4bd - 3ad$
(t) $x^2 + 9x + 20$
(u) $x^2 - 8x + 12$
(v) $b^2 - b - 12$
*(w) $4m^2 - 28m + 40$
*(x) $bc^2 - 5bc - 6b$
*(y) $x - x^5$

3. Find the truth set of each of the following sentences:

(a) $6n = 34$
(b) $3a + 5 = 17$
(c) $16a - 14 = 9a$
(d) $5m + 8 = 4m - 12$
(e) $7x - 3 = 10x$
(f) $\frac{1}{2}x = -7$
(g) $18 + 3(a - 5) = 12$
(h) $3(m + 2) = 4(m - 3)$
(i) $18 - (a - 10) = 0$
(j) $4m > 8$
(k) $\frac{1}{2}n > 3$
(l) $-\frac{2}{3}a > 4$
(m) $3x - 3 < 4$
(n) $4x - 2 > x + 7$
(o) $|x - 2| = 1$
(p) $b^2 - 10b + 21 = 0$
(q) $a^2 - a = 12$
(r) $x^2 - 24 = -2x$
(s) $y^2 = 12y - 36$
(t) $m^2 - 3m = 0$
(u) $a^2 - 16 = 0$
(v) $4b^2 - 36 = 0$

4. Draw the graphs of the truth sets of the sentences in Problem 3(j), (k), (l), (m) .

Review Problem Set
(continued)

5. Translate each of the following into an open sentence and find the truth set:
   - (a) Find two consecutive integers whose sum is 63.
   - (b) One number is five times another and their sum is 90. Find the numbers.
   - (c) James is five years older than George. If the sum of their ages is 37, how old is each?
   - (d) The perimeter of a rectangular lot is 240 feet. Its area is 2700 square feet. Find its length and its width.
   - (e) The area of a triangle is 60 square inches. The base is 16 inches. What is the altitude of the triangle?
   - (f) The sum of three consecutive numbers is 108. What are the numbers?
   - *(g) The sum of two consecutive odd numbers is a three digit number which is less than 110. What are the numbers?
   - *(h) Find two consecutive even numbers (integers) whose sum is 35.

6. On a separate set of axes draw the graph of the truth set of each of the following sentences.
   - (a) $2x + y - 4 = 0$
   - (b) $2x + y - 6 = 0$
   - (c) $5x + y = 0$
   - (d) $2x = 3y$
   - (e) $x + 3y = 6$
   - (f) $-4x + y = -8$
   - (g) $x = 3$
   - (h) $y = -3$
   - (i) $x = 1$
   - (j) $|x| = 2$

7. Write the equation whose graph has
   - (a) slope -3, y-intercept $(0, 2)$
   - (b) slope $\frac{2}{3}$, y-intercept $(0, \frac{3}{2})$
   - (c) slope $-\frac{5}{8}$, y-intercept $(0, 0)$
   - (d) slope 4, y-intercept $(0, 4)$
   - (e) slope -8, y-intercept $(0, -3)$

Review Problem Set
(continued)

8. State the slope and the y-intercept of each of the following sentences.
   (a) $y = 3x - 5$
   (b) $2y + 4x = 2$
   (c) $y + 7x - 5 = 0$
   (d) $3y - 5x + 4 = 6$
   (e) $4x - 2y + 7 = 0$
   (f) $3y - 15 = 0$

Chapter 17

SYSTEMS OF OPEN SENTENCES

17-1. Systems of Equations.

Listed below are some examples of compound open sentences:

$$x + 1 = 5 \quad \text{or} \quad x < 7,$$
$$x + 2 = 8 \quad \text{and} \quad x > 2,$$
$$x^2 < 9 \quad \text{and} \quad x + 3 = 5,$$
$$x^2 < 9 \quad \text{or} \quad x^2 > 9.$$

Some of these compound open sentences involve the connecting word "or;" others involve the connecting word "and." Each of the examples above involves only one variable. However, a compound open sentence may involve two variables. For example,

$$x + y = 5 \quad \text{or} \quad 2x - y = 3\ ,$$
$$2x + y - 1 = 0 \quad \text{and} \quad x + 2y - 5 = 0$$

are examples of compound open sentences in two variables. When such a sentence involves the connecting word "and," it is called a system of sentences. If each of the individual sentences in the system is an equation, it may be called a system of equations.

A system may be written in more than one way. For example, the system

$$2x + y - 1 = 0 \quad \text{and} \quad x + 2y - 5 = 0$$

may be written like this:

$$\begin{cases} 2x + y - 1 = 0 \\ x + 2y - 5 = 0\ . \end{cases}$$

No matter how it is written, a system of sentences is a compound open sentence with connecting word "and."

Recall that a truth number of a compound open sentence with connecting word "and" must satisfy each individual sentence in the compound sentence. Therefore, an element of the truth set of a system of sentences must satisfy each sentence in the system.

Consider again the system of equations $\begin{cases} 2x + y - 1 = 0 \\ x + 2y - 5 = 0\ . \end{cases}$

There are two individual equations in this system. Each of them has infinitely many solutions of the form (x, y). For example, the truth set of "$x + 2y - 5 = 0$" includes

$(5, 0)$, because "$5 + 2(0) - 5 = 0$" is a true sentence;
$(1, 2)$, because "$1 + 2(2) - 5 = 0$" is a true sentence;
$(-1, 3)$, because "$-1 + 2(3) - 5 = 0$" is a true sentence.

However, it is not true that each one of these ordered pairs belongs in the truth set of the system.

In order to be in the truth set of a system, an ordered pair must satisfy all the sentences of the system. In the case above, there are two equations. So any element of the truth set must satisfy both equations.

From this definition, it is easy to see why $(5, 0)$, for example, is not an element of the truth set of the system. It satisfies "$x + 2y - 5 = 0$", but it does not satisfy "$2x + y - 1 = 0$."

"$2(5) + 0 - 1 = 0$" is false.
"$5 + 2(0) - 5 = 0$" is true.

In the same way, it is easy to see that $(-1, 3)$ is an element of the truth set of the system. It satisfies each equation of the system.

"$2(-1) + 3 - 1 = 0$" is true.
"$-1 + 2(3) - 5 = 0$" is true.

Check Your Reading

1. Which of the following sentences may be called a system of sentences?

   (a) $x + y = 5$ or $2x - y = 3$.
   (b) $2x + y - 1 = 0$ and $x + 2y - 5 = 0$.
   (c) $x + y = 5$ or $x < 3$.
   (d) $x + y = 5$ and $x < 3$.
   (e) $x = 6$ and $y = -2$.
   (f) $x = 6$ or $y = -2$.

Check Your Reading
(continued)

2. How do you explain the fact that (0, 1) is *not* an element of the truth set of the system $\begin{cases} 2x + y - 1 = 0 \\ x + 2y - 5 = 0 \end{cases}$?

3. How do you explain the fact that (-1, 3) *is* an element of the truth set of the system $\begin{cases} 2x + y - 1 = 0 \\ x + 2y - 5 = 0 \end{cases}$?

4. What must be true of each element in the truth set of a system of sentences?

Problem Set 17-1

1. In each of the parts of this question, you will find an ordered pair listed together with a system of sentences. In each case, decide whether or not the ordered pair satisfies the system.

   (a) (5, 5); $\begin{cases} x + y = 10 \\ x - y = 10 \end{cases}$

   (b) (1, 3); $2x + y - 5 = 0$ and $3x = y$

   (c) (1, 1); $\begin{cases} 2x + 2y - 4 = 0 \\ x = y \end{cases}$

   (d) (8, 4); $\begin{cases} x + y = 12 \\ y < 2 \end{cases}$

   (e) (8, 4); $\begin{cases} x + y = 12 \\ y = 2 \end{cases}$

   (f) (1, 4); $y = 4$ and $x - y = -3$

   (g) (5, 5) $\begin{cases} 3x - y - 10 = 0 \\ 13x - 3y - 50 = 0 \end{cases}$

   (h) $(\frac{1}{2}, \frac{1}{3})$; $x + y = \frac{5}{6}$ and $x - y = \frac{1}{6}$

   (i) (3, 2); $\begin{cases} x = 3 \\ y = 2 \end{cases}$

Problem Set 17-1
(continued)

2. For which of the following compound sentences is $(9, 1)$ an element of the truth set?

   (a) $x + y = 10$ and $x < 8$

   (b) $x + y = 10$ or $x < 8$

   (c) $\begin{cases} x + y = 10 \\ x > 8 \end{cases}$

   (d) $x + y = 10$ or $x > 8$

   (e) $x = 9$ and $y = 1$

   (f) $\begin{cases} x = 9 \\ y = 1 \end{cases}$

3. List five ordered pairs that satisfy the system "$x + y = 20$ and $x < 10$."

4. List five ordered pairs that do not satisfy the system

   $\begin{cases} x + y = 20 \\ x < 10 \end{cases}$.

5. List five different systems that the element $(4, 2)$ satisfies.

17-2. Graphs of Systems of Equations.

In the previous section, the truth set of a system of equations was defined as the set of elements satisfying all equations of the system. However, no mention was made of a method for determining the truth set, other than by guesswork. Also, no mention was made of how many elements are in the truth set of a system of two first degree equations.

One way to approach such questions as these is to draw the graph of each sentence in the system. For the system

$$\begin{cases} 2x + y - 1 = 0 \\ x + 2y - 5 = 0 \end{cases},$$

we have already seen (in the previous section) that (-1, 3) is an element of the truth set. The graph of each equation in this system has been drawn in Figure 1.

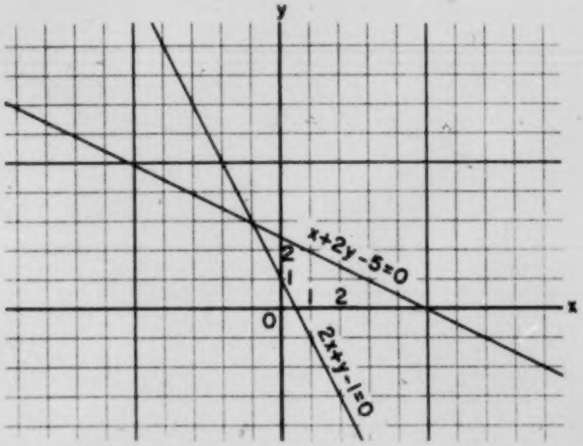


Figure 1

Each of the graphs (as we learned in Chapter 16) is a line. Notice the point of intersection, the point which is on both lines. Two different lines cannot have more than one point in common; that is, there cannot be more than one point of intersection.

This point of intersection is on both graphs; therefore, its coordinates satisfy both equations of the system. Since it is the only point of intersection of the two lines, its coordinates represent the only ordered pair that satisfies both equations. Because we saw in the last section that the ordered pair (-1, 3) satisfies both equations, (-1, 3) must be the coordinates of the point of intersection of the lines. Furthermore, (-1, 3) is the only ordered pair satisfying both equations. There is one and only one element in the truth set of the system.

As another example, consider the problem of finding the truth set of the system

$$\begin{cases} 2x - 3y = 2 \\ x + 2y = 8 \end{cases} .$$

The graph of each equation in the system is shown in Figure 2.

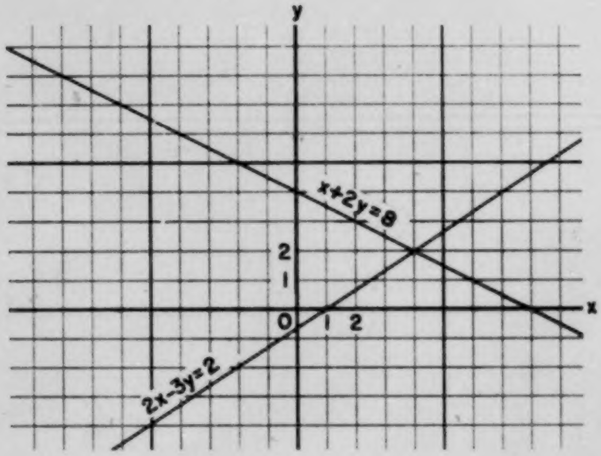


Figure 2

There is one and only one point of intersection of the two lines. From the figure, the coordinates of this point appear to be (4, 2). Because reading coordinates from a figure is approximate at best, this ordered pair should be tested in each sentence of the system.

"$2(4) - 3(2) = 2$" is a true sentence, because each side names the number 2.

"$(4) + 2(2) = 8$" is a true sentence, because each side names the number 8.

Therefore, the truth set of the system $\begin{cases} 2x - 3y = 2 \\ x + 2y = 8 \end{cases}$ is $\{(4, 2)\}$.

In the examples of this section, the truth set of a system of equations was determined from the graphs of the individual equations of the system.

Check Your Reading

1. What kind of figure is the graph of each of the individual sentences in the system

$$\begin{cases} 2x + y - 1 = 0 \\ x + 2y - 5 = 0 \text{ ?} \end{cases}$$

2. What is the greatest number of points in which two different lines can intersect?

3. If the graphs of the individual sentences in the system

$$\begin{cases} 2x - 3y = 2 \\ x + 2y = 8 \end{cases}$$

are drawn, and the graphs intersect in a point, what is true of the coordinates of the point? Why?

4. If the graphs of the individual sentences in the system

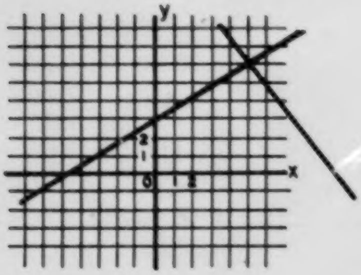
$$\begin{cases} 2x - y - 5 = 0 \\ 3x + 2y + 4 = 0 \end{cases}$$

are two different straight lines that intersect, how many elements are there in the truth set of the system?

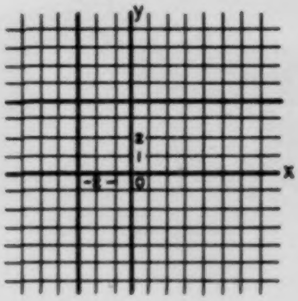
Problem Set 17-2

1. In each of the figures below, two lines have been drawn with reference to a set of axes. Assume in each case that the lines are the graphs of the sentences of a system, and determine the truth set of the system.

(a)

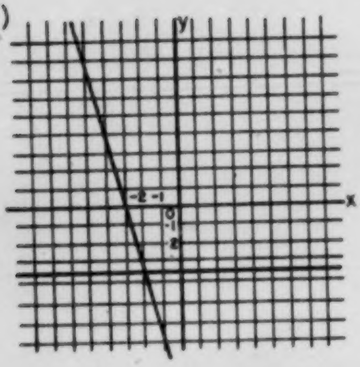


(b)

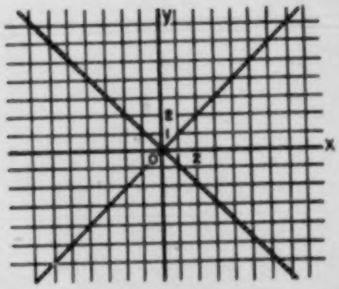


Problem Set 17-2
(continued)

(c)



(d)



2. Determine the truth set of each of the following systems by drawing the graph of each sentence in the system. Because reading coordinates from a figure is only approximate, be sure to test each pair in both sentences of the system.

(a) $\begin{cases} x + y = 5 \\ x - y = 1 \end{cases}$

(b) $x + y = 8$ and $x - y = 12$

(c) $\begin{cases} 2x - y - 2 = 0 \\ 3x + y - 3 = 0 \end{cases}$

(d) $\begin{cases} x = 3 \\ 2x + 3y - 9 = 0 \end{cases}$

(e) $3x - 2y = 0$ and $y - x = 3$

(f) $x - y = 0$ and $x + y = 0$

(g) $\begin{cases} 2x + y = 1 \\ 4x - 2y = 6 \end{cases}$

(h) $\begin{cases} 3x + y + 8 = 0 \\ 4x - 3y + 2 = 0 \end{cases}$

(i) $\begin{cases} x = 4 \\ y = -2 \end{cases}$

Problem Set 17-2
(continued)

3. (a) Draw the graph of the truth set of the sentence "$x + y = 8$ or $x - y = 2$."

   (b) Draw the graph of the truth set of the sentence "$x + y = 8$ and $x - y = 2$."

4. What kind of figure is the graph of the truth set of the system
$$\begin{cases} x + y = 8 \\ x - y = 2 \end{cases}?$$

17-3. Solving Systems of Equations.

In the previous section it was seen that certain systems of equations can be solved by drawing the graph of each equation in the system. This is often a very tedious method, however, and it is difficult to read coordinates from a figure with accuracy. So, in this section, we shall develop a method of solving systems of equations that does not depend on drawing graphs.

Consider again the system
$$\begin{cases} 2x + y - 1 = 0 \\ x + 2y - 5 = 0 \end{cases}$$
whose truth set we have already found to be $\{(-1, 3)\}$. Even though the truth set is already known, we shall try to determine it in a new way that will be useful in future problems and that will lead to an understanding of equivalent systems.

Each of the equations in the system has been written in such a form that one side is a polynomial in x and y, and the other side is the number 0. Let us build a new equation from the given equations and find what connection it might have with the system. First, multiply the polynomial in the first equation by any non-zero number, say 7, getting
$$7(2x + y - 1).$$

Next, multiply the polynomial in the second equation by any non-zero number, say 3, getting

$$3(x + 2y - 5).$$

Next, add these two and form the equation

$$7(2x + y - 1) + 3(x + 2y - 5) = 0.$$

Now we can state the following facts about this equation:

1. The pair $(-1, 3)$, which satisfies the system
$$\begin{cases} 2x + y - 1 = 0 \\ x + 2y - 5 = 0 \end{cases},$$
also satisfies the equation
$$7(2x + y - 1) + 3(x + 2y - 5) = 0. \quad \text{(Why?)}$$

2. Since a point is on the graph of an open sentence if its coordinates satisfy the open sentence, then the point $(-1, 3)$ must be on the graph of
$$7(2x + y - 1) + 3(x + 2y - 5) = 0.$$
In other words, the graph of "$7(2x + y - 1) + 3(x + 2y - 5) = 0$" contains the point $(-1, 3)$, the intersection of the graphs of the equations of the system.

3. The graph of "$7(2x + y - 1) + 3(x + 2y - 5) = 0$" is a line because the equation is equivalent to
$$14x + 7y - 7 + 3x + 6y - 15 = 0,$$
which is equivalent to
$$17x + 13y - 22 = 0,$$
whose graph, as we know from Chapter 16, is a line.

4. By way of summary, then, we can say that the graph of "$7(2x + y - 1) + 3(x + 2y - 5) = 0$" is a line that contains the point of intersection of the graphs of "$2x + y - 1 = 0$" and "$x + 2y - 5 = 0$."
The situation is illustrated in Figure 3.

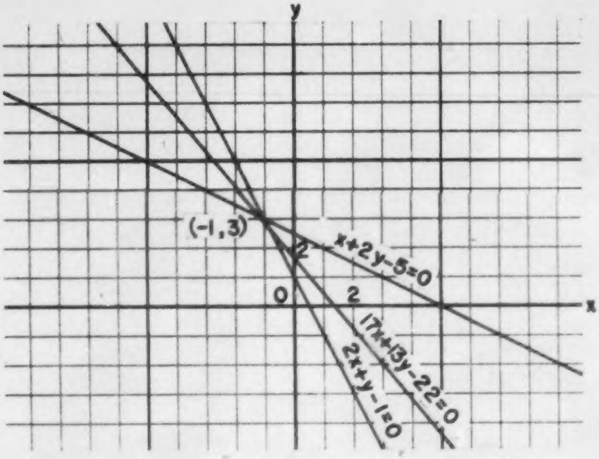


Figure 3

In the system $\begin{cases} 2x + y - 1 = 0 \\ x + 2y - 5 = 0 \end{cases}$,
you may have wondered why the number 7 was chosen as a "multiplier" for the polynomial in the first equation, and 3 was chosen as a "multiplier" for the polynomial in the second equation. The answer is that 7 and 3 were not chosen for any special reason at all. Any non-zero real numbers could have been used.

For example, the graph of each one of the following open sentences is a line containing the intersection of the graphs of "$2x + y - 1 = 0$" and "$x + 2y - 5 = 0$." That is, $(-1, 3)$ satisfies each sentence.

$$4(2x + y - 1) + 6(x + 2y - 5) = 0$$
$$(-2)(2x + y - 1) + (1)(x + 2y - 5) = 0$$
$$(-50)(2x + y - 1) + 100(x + 2y - 5) = 0$$

This list could continue without end. We can say that for any non-zero real numbers a and b, the graph of

$$a(2x + y - 1) + b(x + 2y - 5) = 0$$

is a line passing through the intersection of the graphs of "$2x + y - 1 = 0$" and "$x + 2y - 5 = 0$".

It is not difficult to see that $(-1, 3)$ satisfies the sentence "$a(2x + y - 1) + b(x + 2y - 5) = 0$", no matter what numbers a and b represent.

$$a(2(-1) + 3 - 1) + b\left((-1) + 2(3) - 5\right) = a(0) + b(0) = 0$$

Check Your Reading

1. Is (-1, 3) the solution of the system $\begin{cases} 2x + y - 1 = 0 \\ x + 2y - 5 = 0 \end{cases}$? Why?

2. Is (-1, 3) a solution of "$7(2x + y - 1) + 3(x + 2y - 5) = 0$"? Why or why not?

3. Is (-1, 3) a solution of "$-2(2x + y - 1) + 10(x + 2y - 5) = 0$"? Why or why not?

4. Is (-1, 3) a solution of "$a(2x + y - 1) + b(x + 2y - 5) = 0$" for any non-zero real numbers a and b? Why or why not?

Problem Set 17-3a

1. (-1, 3) is the solution of the system $\begin{cases} 2x + y - 1 = 0 \\ x + 2y - 5 = 0 \end{cases}$.

   For each of the following open sentences, answer "yes" or "no" to the question, "Does the graph of the open sentence contain the point (-1, 3)?"

   (a) $2x + y - 1 = 0$
   (b) $x + 2y - 5 = 0$
   (c) $(2x + y - 1) + (x + 2y - 5) = 0$
   (d) $2(2x + y - 1) + (-3)(x + 2y - 5) = 0$
   (e) $x = -1$
   (f) $y = 3$
   (g) $x = -1$ and $y = 3$
   (h) $\begin{cases} x = -1 \\ y = 3 \end{cases}$

2. For each of the systems below, write three different first degree equations whose truth sets contain the solution of the system.

   Example. $\begin{cases} 3x - 4y + 6 = 0 \\ 2x - 7y - 3 = 0 \end{cases}$ "$2(3x-4y+6) + 3(2x-7y-3) = 0$" is one open sentence whose truth set includes the solution of the system at the left.

Problem Set 17-3a
(continued)

(a) $\begin{cases} x + y - 8 = 0 \\ x - y - 2 = 0 \end{cases}$

(b) $\begin{cases} x + y = 6 \\ x - y = 2 \end{cases}$

(c) $\begin{cases} 2x + 8y - 6 = 0 \\ \frac{1}{2}x - 2y + 10 = 0 \end{cases}$

(d) $\begin{cases} x - 3 = 0 \\ y - 2 = 0 \end{cases}$

(e) $2x + 4y - 7 = 0$ and $5x - 7y = 0$.

3. From the text of this section, we know that the solution (if there is one) of the system $\begin{cases} 2x - y + 5 = 0 \\ x + y - 2 = 0 \end{cases}$ is a solution of the open sentence "$a(2x - y + 5) + b(x + y - 2) = 0$", for any non-zero real numbers a and b. In this case, determine the open sentence that results when a is 1 and b is 1. Also determine the open sentence that results when a is 1 and b is -2.

It is true, as we have seen, that for any non-zero real numbers a and b

$$a(2x + y - 1) + b(x + 2y - 5) = 0$$

is an open sentence whose graph is a line containing the intersection of the graphs of "$2x + y - 1 = 0$" and "$x + 2y - 5 = 0$." However, certain choices of a and b lead to especially useful results. In the case above, let a be -1 and let b be 2.

$$\begin{aligned} (-1)(2x + y - 1) + 2(x + 2y - 5) &= 0 \\ (-2x - y + 1) + (2x + 4y - 10) &= 0 \\ 3y - 9 &= 0 \\ y &= 3 \end{aligned}$$

The graph of "$y = 3$" is a line containing the point $(-1, 3)$ as shown in Figure 4. Notice that the choice of -1 for a and of 2 for b resulted in an equation involving only the variable y. Do you see *why* this choice led to this result?

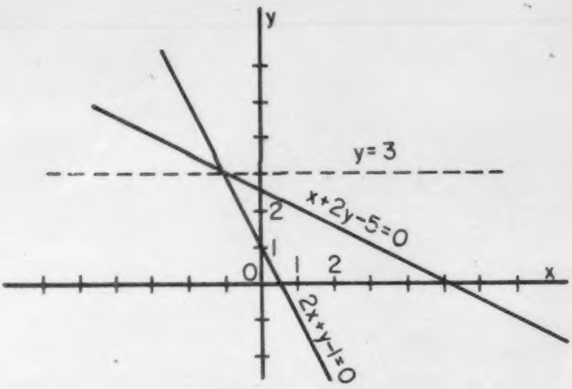


Figure 4

There are choices for a and b which will lead to an equation involving only the variable x. For example, if a is 2 and b is -1, we get

$$
\begin{aligned}
2(2x + y - 1) + (-1)(x + 2y - 5) &= 0 \\
(4x + 2y - 2) + (-x - 2y + 5) &= 0 \\
3x + 3 &= 0 \\
x &= -1
\end{aligned}
$$

The graph of "x = -1" is also a line containing (-1, 3); this is shown in Figure 5.

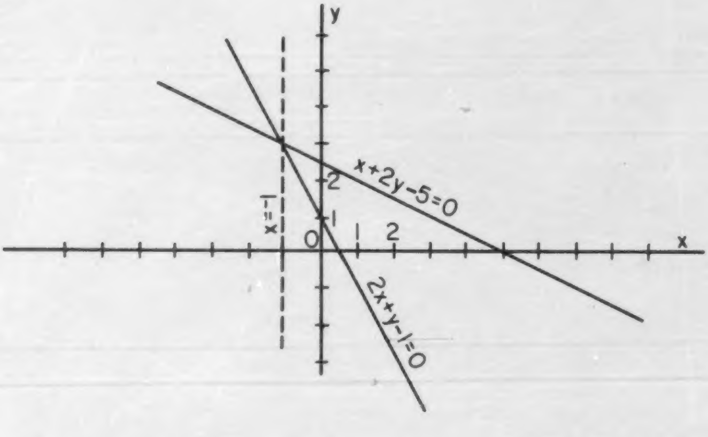


Figure 5

In Figure 6, the graphs of the equations

$2x + y - 1 = 0,\quad x + 2y - 5 = 0,\quad x = -1,\quad y = 3$

are all drawn with respect to the same set of axes.

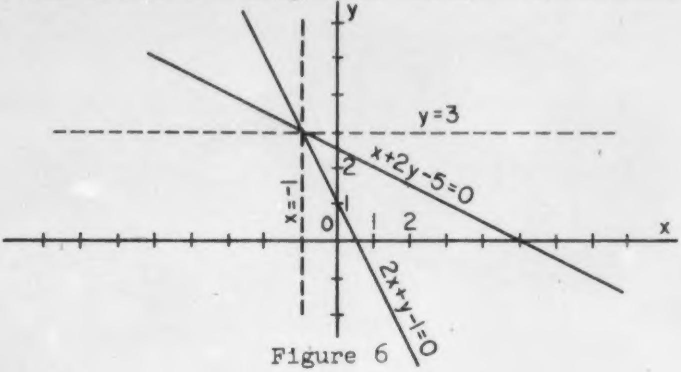


Figure 6

This figure emphasizes the following important fact:

The two systems $\begin{cases} 2x + y - 1 = 0 \\ x + 2y - 5 = 0 \end{cases}$ and $\begin{cases} x = -1 \\ y = 3 \end{cases}$

have the same truth set --- $\{(-1, 3)\}$.

Recall that two sentences with the same truth set are said to be equivalent. Since a system is a compound open sentence with connecting word "and," the two systems

$$\begin{cases} 2x + y - 1 = 0 \\ x + 2y - 5 = 0 \end{cases} \quad \text{and} \quad \begin{cases} x = -1 \\ y = 3 \end{cases}$$

represent two compound open sentences with the same truth set. Therefore, they are called equivalent systems. In general, two systems are equivalent if they have the same truth set.

Notice that one of the two equivalent systems above is much easier to solve than the other. This suggests that a system can be solved by forming an equivalent system whose truth set is quickly determined. The following example illustrates this method.

Example. Find the truth set of the system $\begin{cases} x + 2y + 2 = 0 \\ 4x + 3y - 7 = 0 \end{cases}$.

First, let a be -4 and b be 1.

$$\begin{aligned} -4(x + 2y + 2) + (1)(4x + 3y - 7) &= 0 \\ (-4x - 8y - 8) + (4x + 3y - 7) &= 0 \\ -5y - 15 &= 0 \\ y &= -3 \end{aligned}$$

Do you see why the choice of -4 for a and of 1 for b resulted in an equation involving only the variable y?

Next, let a be 3 and b be -2.

$$\begin{aligned} 3(x + 2y + 2) + (-2)(4x + 3y - 7) &= 0 \\ (3x + 6y + 6) + (-8x - 6y + 14) &= 0 \\ -5x + 20 &= 0 \\ -5x &= -20 \\ x &= 4 \end{aligned}$$

Do you see how the choice of 3 for a and of -2 for b led to the sum $6y + (-6y)$, resulting in an equation involving only the variable x?

The systems $\begin{cases} x + 2y + 2 = 0 \\ 4x + 3y - 7 = 0 \end{cases}$ and $\begin{cases} x = 4 \\ y = -3 \end{cases}$ are equivalent.

The truth set of the system $\begin{cases} x = 4 \\ y = -3 \end{cases}$ is $\{(4, -3)\}$.

Therefore, the truth set of the system $\begin{cases} x + 2y + 2 = 0 \\ 4x + 3y - 7 = 0 \end{cases}$ is $\{(4, -3)\}$.

Do you see that the sentence "$x + 2y + 2 = 0$ and $4x + 3y - 7 = 0$" is true when x is 4 and y is -3 and only when x is 4 and y is -3?

Figure 7 shows the graph of each of the individual equations in the equivalent systems of the above example.

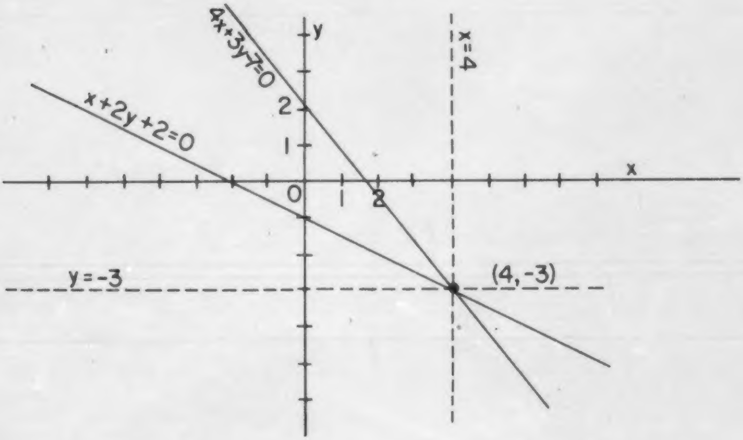


Figure 7

Do you see that the graph of the system $\begin{cases} x + 2y + 2 = 0 \\ 4x + 3y - 7 = 0 \end{cases}$ is the single point with coordinates $(4, -3)$?

Check Your Reading

1. Are the systems $\begin{cases} 2x + y - 1 = 0 \\ x + 2y - 5 = 0 \end{cases}$ and $\begin{cases} x = -1 \\ y = 3 \end{cases}$ equivalent? Why or why not?

2. Are the systems $\begin{cases} x + y = 8 \\ x - y = 4 \end{cases}$ and $\begin{cases} x = 6 \\ y = 2 \end{cases}$ equivalent? Why or why not?

3. What is the definition of equivalent systems?

4. In the explanation of the text, sentences of the form "$a(2x + y - 1) + b(x + 2y - 5) = 0$" were obtained from the system $\begin{cases} 2x + y - 1 = 0 \\ x + 2y - 5 = 0 \end{cases}$.
   One of these sentences involved only the variable x; what values of a and b were used to obtain it? Another of these sentences involved only the variable y; what values of a and b were used to obtain it?

5. Describe the graph of the system "$2x + y - 1 = 0$ and $x + 2y - 5 = 0$."

Oral Exercises 17-3b

For each of the following systems, an equivalent system can be determined in which one of the sentences involves only the variable x, and the other sentence involves only the variable y. In these exercises, tell what values of a and of b can be used to obtain the equation involving only x, and what values can be used to obtain the equation involving only y.

1. $\begin{cases} 2x + y + 5 = 0 \\ x + 3y - 10 = 0 \end{cases}$

2. $\begin{cases} x + y = 14 \\ x - y = 2 \end{cases}$

3. $\begin{cases} 2x - 3y - 22 = 0 \\ x + y - 6 = 0 \end{cases}$

4. $\begin{cases} x + 2y = 1 \\ 2x - y = -23 \end{cases}$

5. $\begin{cases} x - y = 5 \\ 2x + 3y = -20 \end{cases}$

6. $\begin{cases} 2x - y - 5 = 0 \\ x + 2y - 25 = 0 \end{cases}$

Problem Set 17-3b

Determine the truth set of each of the following systems by first determining an equivalent system in which one of the sentences involves only the variable x, and the other sentence involves only the variable y. For the first two problems, draw graphs of each of the sentences in the systems; then give a word description of the graph of the system.

1. $\begin{cases} x + y - 14 = 0 \\ x - y - 22 = 0 \end{cases}$
2. $\begin{cases} 2x + y + 5 = 0 \\ 2x + 3y - 10 = 0 \end{cases}$
3. $\begin{cases} x + 2y = 1 \\ 2x - y = -23 \end{cases}$
4. $\begin{cases} 2x - 3y - 22 = 0 \\ x + y - 6 = 0 \end{cases}$
5. $\begin{cases} x - y = 5 \\ 2x + 3y + 20 = 0 \end{cases}$
6. $\begin{cases} 2x - y - 5 = 0 \\ x + 2y - 25 = 0 \end{cases}$
7. $\begin{cases} 3x - 2y + 7 = 0 \\ 2x - 5y - 10 = 0 \end{cases}$
8. $\begin{cases} y = 2x + 3 \\ 4x = y - 2 \end{cases}$
9. $\begin{cases} x + 3y - 3.5 = 0 \\ 2x + 4y - 3 = 0 \end{cases}$
10. $\frac{3}{2}x + y - 6 = 0$ and $x + \frac{1}{2}y = \frac{5}{2}$

In the previous section, the system $\begin{cases} 2x + y - 1 = 0 \\ x + 2y - 5 = 0 \end{cases}$ was solved by determining the equivalent system

$$\begin{cases} x = -1 \\ y = 3 \end{cases}.$$

This turned out to be a very convenient equivalent system. However, many other equivalent systems could have been used. In fact, it can be shown that for any non-zero real numbers a and b, the system

$$\begin{cases} 2x + y - 1 = 0 \\ x + 2y - 5 = 0 \end{cases} \qquad \text{(system I)}$$

is equivalent to the system

$$\begin{cases} 2x + y - 1 = 0 \\ a(2x + y - 1) + b(x + 2y - 5) = 0. \end{cases} \qquad \text{(system II)}$$

The systems have been labeled "system I" and "system II" for convenience in the proof that the systems are equivalent, which is given below.

To show that the systems are equivalent, two things must be shown:

(1) Any solution of system I is a solution of system II.

(2) Any solution of system II is a solution of system I.

(1) Suppose (r,s) is a solution of system I. Then the following sentences are true:

$$2r + s - 1 = 0$$
$$r + 2s - 5 = 0.$$

Is (r,s) also a solution of system II? It satisfies the first sentence of system II since we know that "$2r + s - 1 = 0$" is true. In the second sentence of system II, if x is r and y is s, the sentence becomes

$$a(2r + s - 1) + b(r + 2s - 5) = 0,$$

the left side of which may be written

$$a(0) + b(0),$$

which is zero no matter what numbers a and b are. Therefore, the sentence is true, and (r,s) does satisfy system II. We may conclude that any solution of system I is a solution of system II.

(2) Suppose (n,t) is a solution of system II. Then the following sentences are true:

$$2n + t - 1 = 0$$
$$a(2n + t - 1) + b(n + 2t - 5) = 0.$$

Is (n,t) also a solution of system I? It satisfies the first sentence of system I since we know that "$2n + t - 1 = 0$" is true. To see that (n,t) also satisfies the second sentence of system I, remember that the sentence $a(2n + t - 1) + b(n + 2t - 5) = 0$ is true. This sentence may be written:

$$a(0) + b(n + 2t - 5) = 0$$
$$0 + b(n + 2t - 5) = 0$$
$$b(n + 2t - 5) = 0$$

> Since b is not zero, this means that $n + 2t - 5$ must be zero. So (n,t) does satisfy the second sentence of system I and is a solution of system I. We may conclude that *any solution of system II is a solution of system I*.

Either of the equations of the original system could have been used in system II. That is, exactly the same kind of argument would have shown the following systems to be equivalent:

$$\begin{cases} 2x + y - 1 = 0 \\ x + 2y - 5 = 0 \end{cases}$$

$$\begin{cases} a(2x + y - 1) + b(x + 2y - 5) = 0 \\ x + 2y - 5 = 0 \end{cases}$$

In fact, we can be much more general than this. Suppose m and n represent two first degree polynomials in x and y. (For example, m might be $3x - 2y + 7$ and n might be $x - 8y - 2$.) Then form the system

$$\begin{cases} m = 0 \\ n = 0. \end{cases}$$

For any non-zero real numbers a and b, the following systems are equivalent to this system:

$$\begin{cases} m = 0 \\ am + bn = 0 \end{cases}$$

$$\begin{cases} am + bn = 0 \\ n = 0. \end{cases}$$

Check Your Reading

Which of the systems below is equivalent to the system

$$\begin{cases} 2x + y - 1 = 0 \\ x + 2y - 5 = 0 \end{cases} ?$$

1. $\begin{cases} 2x + y - 1 = 0 \\ a(2x + y - 1) + b(x + 2y - 5) = 0 \end{cases}$
   $a \neq 0,\ b \neq 0$

2. $\begin{cases} a(2x + y - 1) + b(x + 2y - 5) = 0 \\ x + 2y - 5 = 0 \end{cases}$
   $a \neq 0,\ b \neq 0$

Check Your Reading
(continued)

3. $\begin{cases} 3(2x + y - 1) + 5(x + 2y - 5) = 0 \\ x + 2y - 5 = 0 \end{cases}$

4. $\begin{cases} 2x + y - 1 = 0 \\ \,x = -1 \end{cases}$

5. $2x + y - 5 = 0$ and $y = 3$

6. $\begin{cases} x = -1 \\ y = 3 \end{cases}$

Problem Set 17-3c

1. Form a system equivalent to the system $\begin{cases} 2x - 3y - 4 = 0 \\ x + 2y - 9 = 0 \end{cases}$ by using the first sentence of the system, together with a sentence of the form "$a(2x - 3y - 4) + b(x + 2y - 9) = 0$," where
   (a) $a = -1$ and $b = 2$.
   (b) $a = 2$ and $b = 3$.
   (c) $a = -10$ and $b = 6$.

2. Form a system equivalent to the system $\begin{cases} 2x - 3y - 4 = 0 \\ x + 2y - 9 = 0 \end{cases}$ by using the second sentence of the system, together with a sentence of the form "$a(2x - 3y - 4) + b(x + 2y - 9) = 0$," where
   (a) $a = 1$ and $b = -2$.
   (b) $a = 4$ and $b = 6$.
   (c) $a = 7$ and $b = -9$.

3. Without determining the truth set of the system, form five different systems, each of which is equivalent to the system

$$\begin{cases} 4x - 3y - 8 = 0 \\ 2x + y + 7 = 0 \end{cases}$$

To find the truth set of the system $\begin{cases} y = 7x + 5 \\ 4x = y - 3 \end{cases}$, the addition property of equality may be applied to each of the equations in the system. The system may then be written like this

$$\begin{cases} -7x + y - 5 = 0 \\ 4x - y + 3 = 0 \end{cases}$$

The following equations represent a line which contains the intersection of the lines whose equations are "$-7x + y - 5 = 0$" and "$4x - y + 3 = 0$."

$$\begin{aligned} (1)(-7x + y - 5) + (1)(4x - y + 3) &= 0 \\ (-7x + y - 5) + (4x - y + 3) &= 0 \\ -3x - 2 &= 0 \\ x &= -\frac{2}{3} \end{aligned}$$

The two systems below are equivalent:

$$\begin{cases} -7x + y - 5 = 0 \\ 4x - y + 3 = 0 \end{cases} \quad \text{and} \quad \begin{cases} -7x + y - 5 = 0 \\ x = -\frac{2}{3} \end{cases}$$

The system on the left is the one whose solution is to be found. Since the two systems are equivalent, the system on the right may be considered instead. If there is a solution of the system on the right, it must be of the form

$$\left(-\frac{2}{3}, b\right)$$

since the second equation says that x must be $-\frac{2}{3}$. If there is such a solution, then, from the first equation of the system the number b must satisfy the following:

$$\begin{aligned} -7\left(-\frac{2}{3}\right) + b - 5 &= 0 \\ \frac{14}{3} + b - 5 &= 0 \\ b - \frac{1}{3} &= 0 \\ b &= \frac{1}{3}. \end{aligned}$$

The four sentences above are equivalent ones. Therefore, if b is $\frac{1}{3}$, not only is the last one true, but the first is true also, and a solution of the system has been found.

$$\left(-\frac{2}{3}, \frac{1}{3}\right) \text{ is the solution of the system } \begin{cases} y = 7x + 5 \\ 4x = y - 3 \end{cases}.$$

The method above is essentially the same as that developed in the previous section. The only difference lies in the fact that instead of determining two equations, one involving only x and one involving only y, we determined only one of these, and proceeded directly to the solution. Another example, in which the steps have been shortened, is given below.

Example.

$$\begin{cases} 2x - 5y - 1 = 0 \\ 4x + 2y - 14 = 0 \end{cases}$$

This is the system whose solution is to be found.

$$\begin{cases} 2x - 5y - 1 = 0 \\ -2(2x - 5y - 1) + (4x + 2y - 14) = 0 \end{cases}$$

$$\begin{cases} 2x - 5y - 1 = 0 \\ 12y - 12 = 0 \end{cases}$$

$$\begin{cases} 2x - 5y - 1 = 0 \\ y = 1 \end{cases}$$

This system is equivalent to the original system. Because of the second equation, if there is a solution of the system, it must be of the form $(a, 1)$. If there is such a solution, the number a must satisfy the following equivalent sentences:

$$\begin{aligned} 2(a) - 5(1) - 1 &= 0 \\ 2a - 6 &= 0 \\ a &= 3 \end{aligned}$$

$(3, 1)$ is the solution of the original system.

Check Your Reading

1. If the system $\begin{cases} -7x + y - 5 = 0 \\ x = -\frac{2}{3} \end{cases}$ has a solution, of what form must this solution be?
2. If the system $\begin{cases} 2x - 5y - 1 = 0 \\ y = 1 \end{cases}$ has a solution, of what form must the solution be?
3. If the system $\begin{cases} 3x + 2y - 7 = 0 \\ y = .2 \end{cases}$ has a solution, of what form must it be?
4. If the system $\begin{cases} x - y - 10 = 0 \\ x = 5.7 \end{cases}$ has a solution, of what form must it be?
5. In the explanation in the text of this section, an equation of the form "$a(-7x + y - 5) + b(4x - y + 3) = 0$" was obtained, in which only the variable x was involved. What values of a and b were used?

Problem Set 17-3d

Find the truth set of each of the following systems by first determining an equivalent system in which one of the equations involves only one variable.

1. $\begin{cases} x - y = 1 \\ x + y = 5 \end{cases}$

2. $\begin{cases} x - y + 1 = 0 \\ x - 2y - 2 = 0 \end{cases}$

3. $\begin{cases} 2x - 5y - 7 = 0 \\ 3x + y - 2 = 0 \end{cases}$

4. $4x + y - 1 = 0$ and $x - 2y - 16 = 0$

5. $\begin{cases} 3x = 2y - 19 \\ 4y = 3 - x \end{cases}$

6. $\begin{cases} 2x - 3y - 5 = 0 \\ 5x + y + 13 = 0 \end{cases}$

7. $\begin{cases} 3x + 2y = 1 \\ 2x - 3y = 18 \end{cases}$

8. $2x + y - 9 = 0$ and $3x - y - 1 = 0$

9. $\begin{cases} r + 2s + 4 = 0 \\ 3r - s + 5 = 0 \end{cases}$

10. $\begin{cases} 4c - 3d = 3 \\ 3c - 2d = 4 \end{cases}$

11. (a) How many elements are in the truth set of "$x - y + 1 = 0$ and $x - 2y - 2 = 0$"? (See problem 2 above).

    (b) How many elements are in the truth set of "$x - y + 1 = 0$ or $x - 2y - 2 = 0$"?

17-4. Systems of Equations with Many Solutions.

Consider the system $\begin{cases} 2x + 3y - 10 = 0 \\ 4x + 6y - 20 = 0 \end{cases}$.

We know that for any non-zero real numbers a and b, the following system is equivalent to the given one:

$$\begin{cases} 2x + 3y - 10 = 0 \\ a(2x + 3y - 10) + b(4x + 6y - 20) = 0 \end{cases}.$$

If a is -2 and b is 1, the following equivalent system is obtained:

$$\begin{cases} 2x + 3y - 10 = 0 \\ (-4x - 6y + 20) + (4x + 6y - 20) = 0 \end{cases}.$$

This system can be written

$$\begin{cases} 2x + 3y - 10 = 0 \\ \qquad\qquad 0 = 0 \end{cases}.$$

The system seems somewhat strange because one of the sentences is "$0 = 0$." Remember, however, that a solution of the system makes the sentence

$$2x + 3y - 10 = 0 \quad \text{and} \quad 0 = 0$$

true. Since "$0 = 0$" is true, any (x,y) satisfying "$2x + 3y - 10 = 0$" satisfies the system. There are infinitely many ordered pairs (x,y) satisfying "$2x + 3y - 10 = 0$." Therefore, there are infinitely many ordered pairs satisfying the system

$$\begin{cases} 2x + 3y - 10 = 0 \\ 0 = 0 \end{cases}$$

and infinitely many ordered pairs satisfying the equivalent system

$$\begin{cases} 2x + 3y - 10 = 0 \\ 4x + 6y - 20 = 0 \end{cases}$$

which we were to solve. As examples, $(2, 2)$, $(5, 0)$, and $(\frac{1}{2}, 3)$ are some of the elements of the truth set since they satisfy both sentences of the system.

Study the system $\begin{cases} 2x + 3y - 10 = 0 \\ 4x + 6y - 20 = 0 \end{cases}$ and see if you can discover what made it possible to determine an equivalent system in which one of the sentences was "$0 = 0$." The answer to this problem will enable you to recognize any system of two first degree equations in which there are infinitely many solutions.

In previous problems, the given system had one and only one solution. The graphs of the individual equations of the system were lines that intersected in one and only one point. In this problem the system has infinitely many solutions. What do the graphs look like in this case? In the problem set, you will have a chance to find out.

Check Your Reading

1. In the explanation in the text, the sentence "$0 = 0$" was obtained by using specified values for a and b in the sentence "$a(2x + 3y - 10) + b(4x + 6y - 20) = 0$." What values of a and b were used?

Check Your Reading
(continued)

2. How many elements are in the truth set of "$2x + 3y - 10 = 0$ and $0 = 0$"?

3. How many elements are in the truth set of the system
$$\begin{cases} 2x + 3y - 10 = 0 \\ 0 = 0 \end{cases}?$$

4. How many elements are in the truth set of the system
$$\begin{cases} 2x + 3y - 10 = 0 \\ 4x + 6y - 20 = 0 \end{cases}$$

Problem Set 17-4

1. (a) Draw the graph of each of the sentences in the system
$$\begin{cases} 2x + 3y - 10 = 0 \\ 4x + 6y - 20 = 0 \end{cases}.$$
   (b) Is the sentence "$2x + 3y - 10 = 0$" equivalent to the sentence "$4x + 6y - 20 = 0$"? How can one of the sentences be obtained from the other?
   (c) What is true of the truth sets of equivalent sentences? What is true of the graphs of equivalent sentences?

2. (a) Draw the graph of each sentence in the system
$$\begin{cases} x - 2y + 5 = 0 \\ 3x - 6y + 15 = 0 \end{cases}.$$
   (b) Is the sentence "$x - 2y + 5 = 0$" equivalent to the sentence "$3x - 6y + 15 = 0$"? How can one of the sentences be obtained from the other?
   (c) Determine a system equivalent to $\begin{cases} x - 2y + 5 = 0 \\ 3x - 6y + 15 = 0 \end{cases}$ in which one of the sentences is "$0 = 0$."
   (d) How many elements are in the truth set of the system
$$\begin{cases} x - 2y + 5 = 0 \\ 3x - 6y + 15 = 0 \end{cases}?$$

3. (a) Draw the graph of each sentence in the system
$$\begin{cases} 12x - 4y = 0 \\ 3x - y = 0 \end{cases}.$$
   (b) Is "$12x - 4y = 0$" equivalent to "$3x - y = 0$"? Can one of the sentences be obtained from the other? If so, how?

Problem Set 17-4
(continued)

(c) Is it possible to determine a system equivalent to $\begin{cases} 12x - 4y = 0 \\ 3x - y = 0 \end{cases}$ in which one of the sentences is "0 = 0"?

(d) How many elements are in the truth set of the system

$$\begin{cases} 12x - 4y = 0 \\ 3x - y = 0 \end{cases} ?$$

4. (a) Draw the graph of each sentence in the system

$$\begin{cases} x - y = 3 \\ 2x - y = 6 \end{cases} .$$

(b) Is "x - y = 3" equivalent to "2x - y = 6"? Can one of them be obtained from the other? If so, how?

(c) Is it possible to determine a system equivalent to $\begin{cases} x - y = 3 \\ 2x - y = 6 \end{cases}$ in which one of the sentences is "0 = 0"?

(d) How many elements are in the truth set of the system

$$\begin{cases} x - y = 3 \\ 2x - y = 6 \end{cases} ?$$

5. (a) Describe a way in which it is possible to recognize a system of two linear equations in two variables which has infinitely many solutions.

(b) Describe the graphs of the individual sentences in a system of two linear equations in two variables that has infinitely many solutions.

6. Without determining the truth set and without determining an equivalent system, decide which of the following systems have infinitely many solutions.

(a) $\begin{cases} 2x + 5y - 2 = 0 \\ 4x + 10y - 4 = 0 \end{cases}$

(b) $\begin{cases} x + 7y = 0 \\ \frac{1}{2}x + \frac{7}{2}y = 0 \end{cases}$

(c) $\begin{cases} x + 3y + 12 = 0 \\ 2x + 6y + 6 = 0 \end{cases}$

(d) $\begin{cases} .5x + .5y = 1.5 \\ x + y = 3 \end{cases}$

(e) $\begin{cases} 23x + 18y - 19 = 0 \\ 0 = 0 \end{cases}$

(f) $\begin{cases} 3x + 2y - 5 = 0 \\ 3ax + 2ay - 5a = 0 \\ a \neq 0 \end{cases}$

Problem Set 17-4
(continued)

7. For each of the following systems (where possible), determine an equivalent system in which one of the sentences is "$0 = 0$." If it is not possible, state this.

(a) $\begin{cases} 3x - 5y + 4 = 0 \\ 6x - 10y + 8 = 0 \end{cases}$

(b) $\begin{cases} x - 3y - 1 = 0 \\ -x + 3y + 1 = 0 \end{cases}$

(c) $\begin{cases} x - 7y + 3 = 0 \\ x - 14y + 6 = 0 \end{cases}$

(d) $2x + y - 6 = 0$ and $4x + 2y - 12 = 0$.

17-5. Systems with No Solution.

Consider the system $\begin{cases} 2x + 4y - 10 = 0 \\ 2x + 4y + 5 = 0 \end{cases}$.

For any non-zero real numbers a and b, the following system is equivalent to the given one:

$$\begin{cases} 2x + 4y - 10 = 0 \\ a(2x + 4y - 10) + b(2x + 4y + 5) = 0. \end{cases}$$

If a is 1 and b is -1, the following equivalent system is obtained:

$$\begin{cases} 2x + 4y - 10 = 0 \\ (2x + 4y - 10) + (-2x - 4y - 5) = 0 . \end{cases}$$

This system can be written

$$\begin{cases} 2x + 4y - 10 = 0 \\ \qquad\quad -15 = 0 . \end{cases}$$

A solution of this system must make the compound sentence

$$2x + 4y - 10 = 0 \text{ and } -15 = 0$$

true. Since "$-15 = 0$" is not true, there is no ordered pair (x,y) that will satisfy the system

$$\begin{cases} 2x + 4y - 10 = 0 \\ \qquad\quad -15 = 0 . \end{cases}$$

We must conclude also that the equivalent system

$$\begin{cases} 2x + 4y - 10 = 0 \\ 2x + 4y + 5 = 0 \end{cases}$$

has no solution.

Study the system $\begin{cases} 2x + 4y - 10 = 0 \\ 2x + 4y + 5 = 0 \end{cases}$ and see if you can discover what made it possible to obtain an equivalent system containing the sentence "$-15 = 0$". Why were the values 1 and -1 chosen for a and b? The answers to these questions will help you identify any system of two linear equations that has no solution.

In the problem set, you will have a chance to discover what the graphs of the sentences in a system with no solution look like.

Check Your Reading

1. In the explanation in the text, the sentence "$-15 = 0$" was obtained by using specified values of a and b in the sentence "$a(2x + 4y - 10) + b(2x + 4y + 5) = 0$." What values of a and b were used?
2. How many elements are in the truth set of the sentence "$2x + 4y - 10 = 0$ and $-15 = 0$"?
3. How many elements are in the truth set of the system $\begin{cases} 2x + 4y - 10 = 0 \\ -15 = 0 \end{cases}$?

Problem Set 17-5

1. (a) Draw the graph of each sentence in the system $\begin{cases} 2x + 4y - 10 = 0 \\ 2x + 4y + 5 = 0 \end{cases}$.

   (b) How is the graph of "$2x + 4y - 10 = 0$" related to the graph of "$2x + 4y + 5 = 0$"?

   (c) Is the sentence "$2x + 4y - 10 = 0$" equivalent to the sentence "$2x + 4y + 5 = 0$"? How are the coefficients of x and y in the two sentences related? Are the constant terms related in the same way?

2. (a) Draw the graph of each sentence in the system

$$\begin{cases} x + y - 6 = 0 \\ x + y - 2 = 0 \end{cases} .$$

Problem Set 17-5
(continued)

(b) Is "$x + y - 6 = 0$" equivalent to "$x + y - 2 = 0$"? How are the coefficients of x and y in the two sentences related? Are the constant terms related in the same way?

(c) Determine a system equivalent to $\begin{cases} x + y - 6 = 0 \\ x + y - 2 = 0 \end{cases}$ in which one of the sentences is "$4 = 0$." (Hint: Use -1 for a and 1 for b in the sentence "$a(x + y - 6) + b(x + y - 2) = 0$.")

(d) How many elements are in the truth set of the system $\begin{cases} x + y - 6 = 0 \\ x + y - 2 = 0 \end{cases}$?

3. (a) Draw the graph of each sentence in the system $\begin{cases} 2x + y - 1 = 0 \\ 4x + 2y - 5 = 0 \end{cases}$.

(b) Is "$2x + y - 1 = 0$" equivalent to "$4x + 2y - 5 = 0$"? How are the coefficients of x and y in the two sentences related? Are the constant terms related in the same way?

(c) Determine a system equivalent to $\begin{cases} 2x + y - 1 = 0 \\ 4x + 2y - 5 = 0 \end{cases}$ in which one of the sentences is false.

(d) How many elements are in the truth set of the system $\begin{cases} 2x + y - 1 = 0 \\ 4x + 2y - 5 = 0 \end{cases}$?

4. (a) Draw the graph of each sentence in the system $\begin{cases} 2x - y - 4 = 0 \\ x + y - 6 = 0 \end{cases}$.

(b) Is "$2x - y - 4 = 0$" equivalent to "$x + y - 6 = 0$"? Do the coefficients of x and y in the two sentences seem to be related in a special way?

(c) Determine, if possible, a system equivalent to $\begin{cases} 2x - y - 4 = 0 \\ x + y - 6 = 0 \end{cases}$ such that the coefficients of x and y in one of the sentences are zero.

(d) How many elements are in the truth set of $\begin{cases} 2x - y - 4 = 0 \\ x + y - 6 = 0 \end{cases}$?

Problem Set 17-5
(continued)

5. Describe a way in which a system of two linear equations in two variables having no solution can be recognized. Describe the graphs of the individual sentences in a system of two linear equations in two variables having no solution.

6. Without finding the truth set and without determining an equivalent system, tell whether each of the following systems has one solution, no solutions, or infinitely many solutions.

(a) $\begin{cases} x + 2y - 7 = 0 \\ 2x + 4y - 14 = 0 \end{cases}$ (e) $\begin{cases} x - y = 5 \\ 2x - 2y = 7 \end{cases}$

(b) $\begin{cases} x + 2y - 7 = 0 \\ 2x + 4y + 5 = 0 \end{cases}$ (f) $\begin{cases} 3x + y - 10 = 0 \\ -3x - y + 3 = 0 \end{cases}$

(c) $\begin{cases} x + 2y - 7 = 0 \\ 2x - 3y - 14 = 0 \end{cases}$ (g) $\begin{cases} x + y - 2 = 0 \\ ax + ay - 2a = 0 \end{cases}$ $a \neq 0$

(d) $\begin{cases} 3x - y + 2 = 0 \\ 6x - 2y + 2 = 0 \end{cases}$ (h) $\begin{cases} x + y - 2 = 0 \\ ax + ay - 2b = 0 \end{cases}$ $\begin{matrix} a \neq 0 \\ a \neq b \end{matrix}$

7. For each of the following systems, determine (if possible) an equivalent system in which one of the sentences is false.

(a) $\begin{cases} x + 2y - 5 = 0 \\ 2x + 4y + 3 = 0 \end{cases}$

(b) $\begin{cases} x + 3y + 6 = 0 \\ 2x - y + 4 = C \end{cases}$

(c) $2x + y - 3 = 0$ and $6x + 3y - 5 = 0$

17-6. Another Method for Solving Systems.

As we have seen in earlier sections of this chapter, any system of two equations in two variables may be solved by forming an equivalent system whose solution may be determined at sight. It was also found that some such systems have exactly one solution, others have infinitely many solutions, and others have no solution.

There are other methods for solving such systems. One of these, often called the "substitution method," is especially convenient in some cases and is discussed in the examples following.

Example 1. Solve the system $\begin{cases} y = 3x + 4 \\ 2x = y - 3 \end{cases}$.

Perhaps from your experience in previous sections, you are able to tell whether this system has one, many, or no solutions. However, even without knowing that there is a solution, we can say that, if there is a solution, certain statements must be true.

If a is a number and b is a number such that the ordered pair (a,b) is a solution of the above system, then the following sentences must be true:

$$b = 3a + 4,$$
$$2a = b - 3.$$

From the first of these sentences, it can be seen that "$3a + 4$" is a name for the number b. By adding 3 to both sides of the second sentence, it can be seen that "$2a + 3$" is also a name for the number b. In other words, if (a,b) is a solution of the system, then "$3a + 4$" and "$2a + 3$" must be names for the same number. That is,

$$3a + 4 = 2a + 3$$
$$a = -1 .$$

Thus, it has been shown that if (a, b) is a solution of the system, then a must be -1. Since "$3a + 4$" and "$2a + 3$" are both names for the number b, either of them may be used to find the value of b if a is -1. Using the phrase "$3a + 4$,"

$$\begin{aligned} b &= 3a + 4 \\ &= 3(-1) + 4 \\ &= 1 . \end{aligned}$$

So, if (a, b) is a solution of the system, then a must be -1, and b must be 1. That is, if there is a solution, it must be $(-1, 1)$. Since it is possible, however, that there is no solution at all, the pair $(-1, 1)$ should be checked in each sentence of the system, as follows:

$y = 3x + 4$ "$1 = 3(-1) + 4$" is true.

$2x = y - 3$ "$2(-1) = 1 - 3$" is true.

Therefore, $(-1, 1)$ is the solution of the system.

Another example is discussed below. And you will notice in this case that one name for the number b has actually been substituted for another. This is the reason for the name "substitution method," mentioned earlier.

Example 2. Solve the system $\begin{cases} 2x = y - 7 \\ x + 2y - 4 = 0 \end{cases}$.

If there is a solution (a, b) of this system, then the following sentences must be true:

$2a = b - 7$ $a + 2b - 4 = 0$	In each sentence of the system, x has been assigned the value a, and y has been assigned the value b.
$b = 2a + 7$	This sentence is equivalent to "$2a = b - 7$," by the addition property of equality. Notice from this sentence that if (a, b) is a solution of the system, then "$2a + 7$" is another name for the number b.
$a + 2b - 4 = 0$ $a + 2(2a + 7) - 4 = 0$ $a + 4a + 14 - 4 = 0$ $5a + 10 = 0$ $5a = -10$ $a = -2$	The sentence "$a + 2b - 4 = 0$" must be true if (a, b) is a solution of the system. Also, "$2a + 7$" must be a name for the number b. So, the name "$2a + 7$" has been "put in," or substituted, for "b" in the sentence "$a + 2b - 4 = 0$."

Thus, if (a, b) is a solution, a must be -2. The phrase "$2a + 7$" may be used to find the corresponding value of b, as follows:

If a is -2, $\quad 2a + 7 = 2(-2) + 7$

$$\begin{aligned} &= -4 + 7 \\ &= 3. \end{aligned}$$

Therefore, if there is a solution, it is the ordered pair $(-2, 3)$. This pair should be checked in each sentence of the original system.

Below is a third example. You should be able to explain each step.

Example 3. Solve the system $\begin{cases} 2x + y = -9 \\ 3x - 2y + 17 = 0. \end{cases}$

If (a, b) is a solution, then the following sentences are true:

$$2a + b = -9$$
$$3a - 2b + 17 = 0$$

$$b = -9 - 2a$$

$$\begin{aligned} 3a - 2(-9 - 2a) + 17 &= 0 \\ 3a + 18 + 4a + 17 &= 0 \\ 7a + 35 &= 0 \\ a &= -5 \end{aligned}$$

If a is -5, $\quad b = -9 - 2a$

$$\begin{aligned} &= -9 - (-10) \\ &= 1. \end{aligned}$$

Therefore, if there is a solution, it is $(-5, 1)$.

Finally, here is a fourth example.

Example 4. Solve the system $\begin{cases} x + 3y = 7 \\ y = -\frac{1}{3}x + 2. \end{cases}$

If there is a solution (a, b) of this system, then the following sentences must be true:

$$a + 3b = 7$$
$$b = -\frac{1}{3}a + 2.$$

From the second of these sentences, it can be seen that "$-\frac{1}{3}a + 2$" is a name for the number b. Substituting this name for "b" in the first sentence, we have

$$\begin{aligned} a + 3b &= 7 \\ a + 3(-\tfrac{1}{3}a + 2) &= 7 \\ a - a + 6 &= 7 \\ 6 &= 7. \end{aligned}$$

The result "$6 = 7$" does not mean that we have made a mistake! What we have shown is that if there is a solution (a, b), then it must be true that $6 = 7$. Since we recognize "$6 = 7$" as a false sentence, we conclude that there is no solution of the system.

Check Your Reading

1. If (a, b) is a solution of the system $\begin{cases} y = 3x + 4 \\ 2x = y - 3 \end{cases}$, what two sentences must be true?
2. What name for the number b is derived from the sentence "$2a = b - 3$"?
3. What is the result of substituting the name "$2a + 7$" for "b" in the sentence "$a + 2b - 4 = 0$"?

Problem Set 17-6

Solve the following systems of equations using the substitution method.

1. $\begin{cases} y = 2x - 1 \\ y = -x + 2 \end{cases}$
2. $\begin{cases} x - 3y + 8 = 0 \\ y = 3x \end{cases}$
3. $\begin{cases} 4x + y = 3 \\ y = -x \end{cases}$
4. $\begin{cases} 2x + y = 8 \\ y = x - 10 \end{cases}$
5. $\begin{cases} x - 2y + 7 = 0 \\ y = x + 1 \end{cases}$
6. $\begin{cases} 5x + y + 15 = 0 \\ x - y = 3 \end{cases}$
7. $\begin{cases} 2x - 3y = 2 \\ x - 2y + 2 = 0 \end{cases}$
8. $\begin{cases} 4x - 3y + 5 = 0 \\ 3x - 2y = 0 \end{cases}$

Problem Set 17-6
(continued)

9. $\begin{cases} 2x + 3y = 13 \\ 3x - 5y + 9 = 0 \end{cases}$

10. $\begin{cases} 4x + 5y = 5 \\ 6x + 7y = 7 \end{cases}$

17-7. Word Problems.

We have seen before that word problems can often be translated into open sentences in algebra The truth set of the open sentence then leads to an answer to the problem.

Many times it is easier to use two variables in the translation from words to open sentences. Below are two examples in which two variables have been used in the translation. You will notice that when two variables are used a system of two equations is needed.

Example 1. The sum of two positive integers is 7. The difference between the integers is 3. What are the integers?

Let x be one of the integers, say the larger one. Then let y be the smaller integer.

$x + y = 7$ This open sentence says that the sum of the numbers is 7.

$x - y = 3$ This open sentence says that the difference between the numbers is 3. Notice that we subtracted the smaller from the larger.

Solve the system

$\begin{cases} x + y = 7 \\ x - y = 3. \end{cases}$ We are looking for two positive integers whose sum is 7 and whose difference is 3.

Writing the system as

$$\begin{cases} x + y - 7 = 0 \\ x - y - 3 = 0, \end{cases}$$

an equivalent system is

$$\begin{cases} x + y - 7 = 0 \\ (x + y - 7) + (x - y - 3) = 0, \end{cases}$$

which can be written

$$\begin{cases} x + y - 7 = 0 \\ \quad\quad\;\; x = 5. \end{cases} \qquad \text{(Why?)}$$

In the first equation, if x is 5, we have

$$(5) + y = 7,$$
$$y = 2.$$

Thus, an equivalent system is

$$\begin{cases} x = 5 \\ y = 2. \end{cases}$$

So, the truth set of the system is

$$\{(5,2)\}.$$

The numbers 5 and 2 give us an answer to the word problem we started with. Both of them are positive integers. Their sum is 7. Their difference is 3.

Example 2. A candy store is going to make a 40-pound mixture of fruit centers and nut centers. The fruit centers sell at $1.00 per pound; the nut centers sell at $1.40 per pound. In order to make the mixture worth $1.10 per pound, how many pounds of each kind of candy should be used?

Let f be the number of pounds of fruit centers.
Let n be the number of pounds of nut centers.

$$f + n = 40$$

This open sentence says that the number of pounds of fruit centers and the number of pounds of nut centers together is 40. 40 is the weight of the mixture.

The money value of the mixture is 44 dollars.

There are 40 pounds in the mixture, and each pound is worth $1.10. $(40)(\$1.10) = \$44$.

The value of the fruit centers in the mixture is f dollars.

There are f pounds of fruit centers, and each pound is worth $1. $(f)(1) = f$.

The value of the nut centers in the mixture is 1.40n dollars.

There are n pounds of nut centers, and each pound is worth \$1.40. (n)(1.40) = 1.40n.

$f + 1.40n = 44$

This open sentence says that the value of the fruit centers added to the value of the nut centers gives the value of the entire mixture.

Solve the system

$$\begin{cases} f + n = 40 \\ f + 1.40n = 44 \end{cases}$$

We are looking for numbers f and n so that there are 40 pounds in the entire mixture and the value of the mixture is 44 dollars.

Write the system as

$$\begin{cases} f + n - 40 = 0 \\ f + 1.40n - 44 = 0. \end{cases}$$

This is equivalent to

$$\begin{cases} f + n - 40 = 0 \\ -(f + n - 40) + (f + 1.40n - 44) = 0, \end{cases}$$

which can be written

$$\begin{cases} f + n - 40 = 0 \\ \qquad\quad n = 10. \end{cases} \qquad \text{(Why?)}$$

If there is a solution of the system, n must be 10. In the first equation, if n is 10,

$$\begin{cases} f + (10) = 40 \\ \qquad\quad f = 30 \end{cases}$$

Thus, an equivalent system is

$$\begin{cases} f = 30 \\ n = 10 \end{cases}$$

We now have an answer to the word problem. The candy store should use 10 pounds of nut centers and 30 pounds of fruit centers.

Problem Set 17-7

1. Find two numbers whose sum is -23, and whose difference is 7.

Problem Set 17-7
(continued)

2. Find two numbers whose difference is 16 and whose sum is 30.

3. A boy has 10 coins totaling $1.60, which is made up of dimes and quarters. How many dimes and how many quarters has he?

4. A man cashed a check for $500, asking the teller to give him the amount in 5 and 10 dollar bills. If there were 70 bills in all, how many 5 dollar bills did he receive?

5. In a two digit number the unit's digit is twice the ten's digit while five times the unit's digit is 6 less than the given number. Find the number.

6. A man bought 30 pounds of nuts, some at 35 cents a pound, and the rest at 50 cents a pound. How many pounds of each kind did he buy if the mixture cost him 45 cents a pound?

7. One day Mr. Brown employed five men and three boys for $68. The next day he employed three men and five boys for $60. How much did he pay each man and each boy for one day's work?

8. How many pounds of seed at $1.05 a pound may be mixed with seed worth $.85 a pound to give 200 pounds worth $.90 a pound?

9. Mr. Raynelle invested $10,000. He invested part at 5% and the balance at 6%. If his yearly income was $548, how much did he invest at 5%? How much did he invest at 6%?

10. The course of an enemy submarine as plotted on a set of axes can be given by the equation $2x + 3y = 9$. On the same axes a destroyer's course is indicated by the graph of $x - y = 2$. At what point do the paths of the destroyer and submarine intersect?

17-8. Systems of Inequalities.

Systems of equations are not the only systems that are important in mathematics. The open sentences in a system may be inequalities as well as equations. For example the compound open sentence

$$x + 2y - 4 > 0 \qquad \text{and} \qquad 2x - y - 3 > 0$$

is a system of inequalities in two variables. It is often written like this:

$$\begin{cases} x + 2y - 4 > 0 \\ 2x - y - 3 > 0. \end{cases}$$

Just as with a system of equations, a solution of a system of inequalities is an ordered pair that satisfies both inequalities of the system.

To see what the truth set of the above system is, first draw the graph of "$x + 2y - 4 > 0$". The graph is shown in Figure 8.

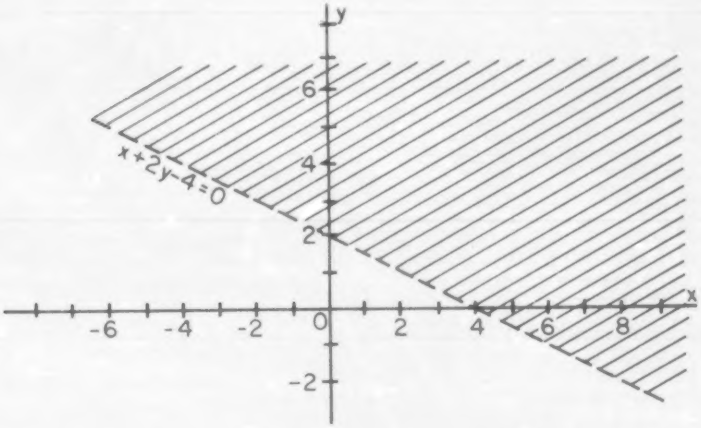


Figure 8

From Chapter 16, we know that the set of points "above" the line "$x + 2y - 4 = 0$" represents ordered pairs that satisfy the inequality "$x + 2y - 4 > 0$". Of course, there are an infinite number of these pairs; so we cannot list them. But at least the graph gives us a kind of "picture" of the truth set of the inequality. (Notice that the line "$x + 2y - 4 = 0$" is dotted, since we do not want to include points on the line.)

Next look at the graph of the second inequality in the system--"$2x - y - 3 > 0$." The graph is shown in Figure 9.

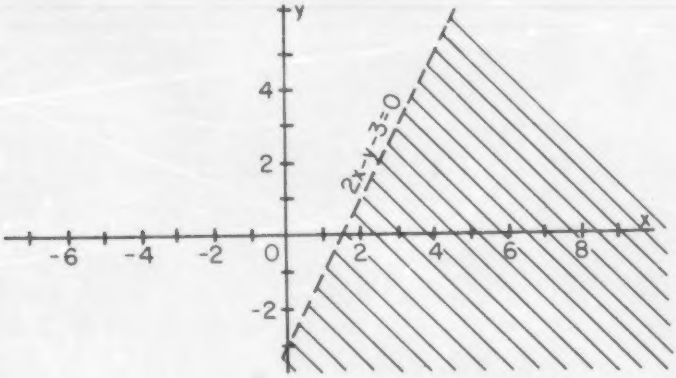


Figure 9

The set of points "below" the line "$2x - y - 3 = 0$" represents ordered pairs which satisfy the inequality "$2x - y - 3 > 0$". Again, there are infinitely many of them. The graph just helps us to get a picture of them.

The two graphs above did not represent a new idea; we graphed inequalities in Chapter 16. But remember that we are looking for the truth set of the system

$$\begin{cases} x + 2y - 4 > 0 \\ 2x - y - 3 > 0. \end{cases}$$

To be in the truth set of the system, an ordered pair must be in the truth set of both inequalities of the system. Let's try drawing the graphs of the two inequalities on the same set of axes. This is shown in Figure 10.

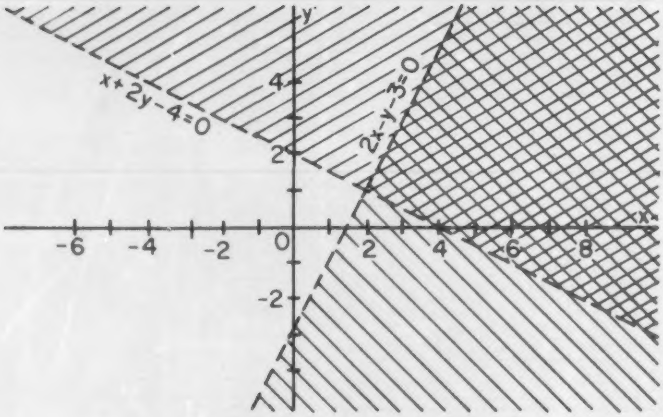


Figure 10

This graph gives us a picture of the truth set of the system. Notice that the graph of one inequality is shown by a shaded region with lines running in one direction; the graph of the other inequality is shown by a shaded region with lines running in another direction. The region with "criss-cross" shading (lines running in both directions) shows points that belong to both graphs. All the points in this criss-cross region represent ordered pairs that satisfy both inequalities. Therefore, this region is a graph of the truth set of the system.

There are infinitely many ordered pairs in the truth set. We cannot list them, and so we must be satisfied with showing the graph of the truth set.

In the system $\begin{cases} 3x - 2y - 5 = 0 \\ x + 3y - 9 \leq 0, \end{cases}$

one of the open sentences is an equation, and one is an inequality. The truth set of the system can be shown graphically.

The graph of "$3x - 2y - 5 = 0$" is shown in Figure 11. Every point on the line represents an ordered pair which satisfies the equation.

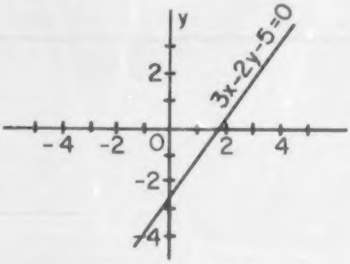


Figure 11

The graph of "$x + 3y - 9 \le 0$" is shown in Figure 12. Every point on the line "$x + 3y - 9 = 0$" and every point below the line represents an ordered pair which satisfies the inequality.

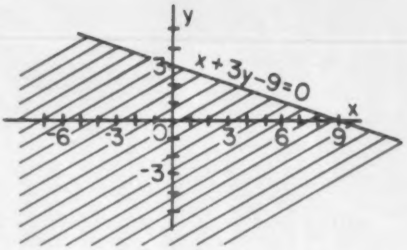


Figure 12

The graph of the system
$$\begin{cases} 3x - 2y - 5 = 0 \\ x + 3y - 9 \le 0 \end{cases}$$
can be shown by drawing the graph of "$3x - 2y - 5 = 0$" and "$x + 3y - 9 \le 0$" on the same set of axes. The graph is shown in Figure 13.

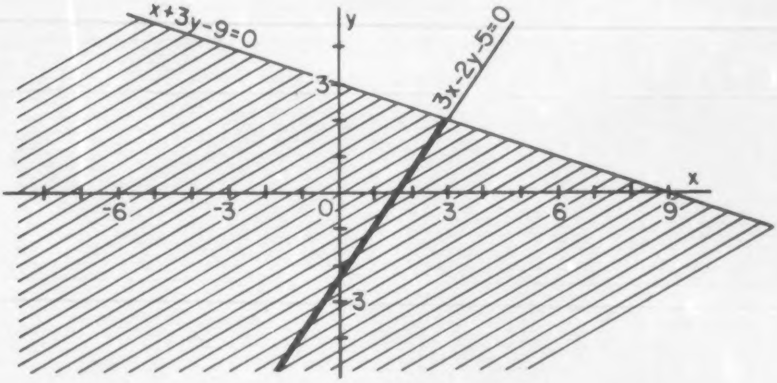


Figure 13

The graph of the system is the set of points that belong to both of the graphs of the individual sentences of the system. In the figure, this is shown as the "darkened" part of the line "$3x - 2y - 5 = 0$".

Problem Set 17-8

For each of the following systems, draw the graph of each sentence in the system, and indicate the truth set of the system by appropriate shading.

1. $\begin{cases} x + y < 9 \\ x - y > 4 \end{cases}$

2. $\begin{cases} 2x + y < 10 \\ x - y \geq 5 \end{cases}$

3. $\begin{cases} 2x + y > 8 \\ 4x - 2y \leq 4 \end{cases}$

4. $\begin{cases} 6x + 3y < 0 \\ 4x - y < 6 \end{cases}$

5. $\begin{cases} 5x + 2y + 1 > 0 \\ 3x - y - 6 = 0 \end{cases}$

6. $\begin{cases} 4x + 2y + 1 = 0 \\ -x + y - 4 \geq 0 \end{cases}$

7. $\begin{cases} 2x + y < 4 \\ 2x + y > 6 \end{cases}$

8. $\begin{cases} 2x + y > 4 \\ 2x + y < 6 \end{cases}$

9. $\begin{cases} 2x - y - 4 \leq 0 \\ 4x - 2y - 8 \leq 0 \end{cases}$

10. $x + y < 8$ and $x > 2$

11. $x + y < 8$ or $x > 2$

Summary

1. A compound open sentence with connecting word "and" is called a system of sentences. The sentences may be equations or inequalities.

2. The truth set of a system of sentences in x and y is the set of all ordered pairs (x, y) that satisfy all of the sentences of the system.

3. Systems with the same truth set are called equivalent systems.

4. A system of two first degree equations in two variables may have:
   (a) exactly one solution, in which case the graph of the system is the single point of intersection of the graphs of the two equations;

(b) no solutions, in which case the graph of the system contains no points because the graphs of the sentences of the system are two line that do not intersect;

(c) infinitely many solutions, in which case the graph of the system is the set of all points on the one line which is the coinciding graph of each of the sentences.

5. The truth set of a system of inequalities may be shown by drawing the graph of each sentence of the system with reference to the same set of axes and shading the appropriate regions.

6. Systems of open sentences furnish a mathematical model for the solution of many word problems.

Review Problem Set

1. Which of the following can be called a system of sentences?

   (a) $3x + 2y = 7$ and $x - 3y = 5$

   (b) $x + y = 5$ or $x - y = 3$

   (c) $(x - 2y + 3)(2x + y - 8) = 0$

   (d) $\begin{cases} x - 3y + 7 = 0 \\ 7x - y = 3 \end{cases}$

   (e) $3 < x < 7$

   (f) $x \geq 5$

2. Given: the system of equations

$$\begin{cases} 3x - y = 3 \\ x - 3y + 7 = 0 \end{cases}$$

   (a) Verify that the truth set is $\{(2, 3)\}$.

   (b) Give a number pair which satisfies each of the following:

$$(3x - y - 3) + 2(x - 3y + 7) = 0$$
$$5(3x - y - 3) + (-3)(x - 3y + 7) = 0$$
$$12(3x - y - 3) + (x - 3y + 7) = 0$$

   (c) Find a and b so that the graph of

$$a(3x - y - 3) + b(x - 3y + 7) = 0$$

   is a horizontal line.

(d) Find a and b so that the graph of

$$a(3x - y - 3) + b(x - 3y + 7) = 0$$

is a vertical line.

(e) Find a and b so that the graph of

$$a(3x - y - 3) + b(x - 3y + 7) = 0$$

contains the origin. Hint: If the graph of

$ax + by + c = 0$ contains the origin,

then $c = 0$.

3. Find the truth set of each of the following open sentences and systems of equations.

(a) $|x - 4| = 0$

(b) $\frac{3}{4}x - 2 < x + 7$

(c) $7x - \frac{3}{4} + x = 8 - 3x + \frac{1}{3}$

(d) $(x - 7)(x + 2)(x - \frac{3}{4}) = 0$

(e) $3x^2 + 15x + 18 = 0$

(f) $x + y - 2 \geq 0$

(g) $\begin{cases} x - 2y = 0 \\ x + 2y = 0 \end{cases}$

(h) $\begin{cases} 5x + 2y - 4 = 0 \\ 3x - 2y - 4 = 0 \end{cases}$

(i) $\begin{cases} x + 2y + 7 = 0 \\ 3x + 6y + 21 = 3 \end{cases}$

(j) $\begin{cases} 5x = 4 - 2y \\ 4y = 8 - 10x \end{cases}$

4. Draw the graphs of the truth sets of Problem 3 (a), (b), and (e) on the number line; and (f), (g), (i), and (j) on the plane.

5. Find the truth set of "$|y| < 3$" and draw its graph if it is considered as a sentence in

(a) one variable (b) two variables.

6. Complete this statement: To every number pair there corresponds exactly ______ ______ in the plane and to every point in the plane there corresponds exactly ______ ______ ______ for a given set of axes in the plane. This is an example of a one-to-one correspondence.

7. Draw the graph of the truth set of each of the systems which follow:

(a) $\begin{cases} x - y - 1 \leq 0 \\ x + 2y \leq 2 \end{cases}$

(b) $\begin{cases} x + 3y - 9 > 0 \\ 5x - 4y = 10 \end{cases}$

Review Problem Set
(continued)

8. Refer to the polynomial $4x^3 - 2x^2 + x + 7$ to answer each of the following questions:

   (a) Which is the second degree term?
   (b) What is the degree of the polynomial?
   (c) What is the coefficient of the first degree term?
   (d) What is the constant term?
   (e) What is the coefficient of the third degree term?
   (f) What is the exponent of the first degree term?
   (g) What is the coefficient of the fourth degree term?
   (h) If the polynomial is set equal to zero do we have a polynomial equation?
   (i) Does (-1) satisfy the equation in part (h) of this problem?

9. Refer to the number pair $(-3, -1)$ to answer each of the following questions:

   (a) What is the x coordinate?
   (b) What is the ordinate?
   (c) In what quadrant does the point lie that corresponds to the number pair?
   (d) What is the opposite of the abscissa?
   (e) In what quadrant do we find the point with the same abscissa and the opposite ordinate?

10. Simplify each of the following expressions.

   (a) $7x(2^2 \cdot 7 \cdot x^3)$
   (b) $(11a + 5b) - (18a - 13b)$
   (c) $\dfrac{24a^2y^3}{32ay^4} \cdot \dfrac{2a^3y}{3a^2}$
   (d) $\dfrac{(12ab^2)^2}{72ab^3}$
   (e) $\dfrac{3x}{2} + \dfrac{2x}{15} - \dfrac{7x}{5}$
   (f) $(-\sqrt{81})^2$
   (g) $(\sqrt[3]{27a})^3$
   (h) $\sqrt{(x + 5)^2}$
   (i) $2a + \dfrac{3}{a} + \dfrac{7}{3a^2}$
   (j) $\dfrac{2}{3ab} - \dfrac{5}{6bc}$

Review Problem Set
(continued)

11. When a tree grows it increases its radius each year by adding a ring of new wood. This ring is made up of dense wood laid down when the tree is growing slowly and light wood when it is growing rapidly. A certain tree increased its radius by 50 millimeters during 365 days. During rapid growth it adds wood at the rate of .25 millimeters per day. During slow growth it adds wood at .10 millimeters per day. How many days of rapid and slow growth did the tree undergo?

12. Two neighboring populations of snails differ in the proportion of red shelled individuals. Population A has 90 % red while B has only 75 % red. Some migrants from these populations meet and form a new colony with 78 % red shells. If the new colony has a total of 200 snails in it, how many snails were contributed by population A and population B to the new colony?

13. A sample of plant tissue was weighed fresh and then the water was completely evaporated and the dry material weighed again. It was found that the dry weight was $\frac{2}{27}$ of the fresh weight and the difference between fresh weight and dry weight was 100 grams. What was the fresh weight?

14. An experimenter weighed a batch of flies containing 2 females and 8 males. The average weight of the 10 flies was 720 micrograms each. When weighed separately the females weighed 940 and 980 micrograms each. What is the average weight of the 8 males?

Chapter 18

QUADRATIC POLYNOMIALS

18-1. Graphs of Quadratic Polynomials.

In Chapter 13 it was said that polynomials such as

$$x^2 - 8x - 3, \quad 2x^2 + 5, \quad 3x^2 - 10x, \quad \frac{1}{2}x^2$$

are called quadratic polynomials in x, because each contains a term of second degree and none of higher degree. Any polynomial in x which can be simplified to this form is called quadratic. For example, "$2x(x - 5) + 3$" is a quadratic polynomial because it can be simplified to the form "$2x^2 - 10x + 3$" in which a second degree term and no higher degree term is present.

Compare the four polynomials listed above. They are all quadratic polynomials in x. What makes any one different from the others? The form is the same, but the coefficients of the terms differ. A specific quadratic polynomial is determined by the coefficients of its three terms. Thus, if we agree to list the coefficients in the order of the degrees of the terms, the three numbers $1, -8, -3$ determine the first quadratic polynomial above, "$x^2 - 8x - 3$". The three numbers $2, 0, 5$ determine the second quadratic polynomial above, "$2x^2 + 5$". What three numbers determine the third quadratic polynomial above? The fourth?

In general, any three real numbers A, B, C, with $A \neq 0$, determine a quadratic polynomial in x,

$$Ax^2 + Bx + C,$$

and any quadratic polynomial in x can be simplified to this form.

Oral Exercises 18-1a

In each of the following decide whether or not the phrase is a quadratic polynomial in x. If the phrase is a quadratic polynomial in x, give the values of the coefficients A, B, and C of the form $Ax^2 + Bx + C$.

1. $2x^2 + 4x + 5$
2. $x^2 + 6x + 8$
3. $\frac{1}{2}x^2 - 6x - 8$
4. $-3x^2 + 5x - \frac{1}{4}$
5. $-x^2 + 6x + \sqrt{2}$
6. $48x^2 + 8x$
7. $-x^2 + 14$
8. $x + 8$
9. $x + 8 - x^2$
10. $5x^3 + 2x^2 - 6x + 4$
11. $5x^3 - 6x + 4$
12. x^2
13. $(x + 1)^2 + 5$
14. $x^2 + tx + k$

The graph of the quadratic polynomial in x, $Ax^2 + Bx + C$, is the graph of the open sentence

$$y = Ax^2 + Bx + C ,$$

where y represents the value of the polynomial for any real number x. That is, the graph is the set of all points (x, y) in a plane, where x is any real number and y is the corresponding value of the polynomial $Ax^2 + Bx + C$.

Example. Draw the graph of $x^2 - 2x - 3$.

A number of ordered pairs (x, y) which satisfy the open sentence

$$y = x^2 - 2x - 3$$

are indicated in the table at the right. In several cases, the value of the polynomial has been left for you to determine.

(x, y)	(x, y)
$(-2, 5)$	$(1, \quad)$
$(-\frac{3}{2}, \frac{9}{4})$	$(\frac{3}{2}, -\frac{15}{4})$
$(-1, \quad)$	$(2, \quad)$
$(-\frac{1}{2}, -\frac{7}{4})$	$(\frac{5}{2}, \quad)$
$(0, \quad)$	$(3, 0)$
$(\frac{1}{2}, \quad)$	$(4, \quad)$

The points corresponding to the ordered pairs (x, y) in the preceding table have been plotted in Figure 1. Can you guess where other points of the graph might be?

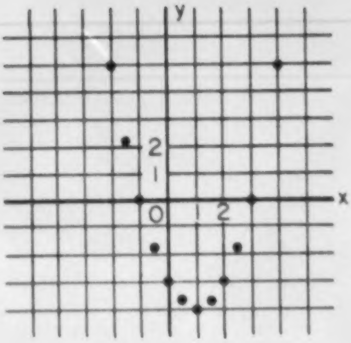


Figure 1.

In Figure 2, a smooth curve has been drawn, containing all of the plotted points. It seems reasonable to say that this curve is a part of the graph of $x^2 - 2x - 3$. A more systematic discussion of the "shape" of such a graph will be found in later work.

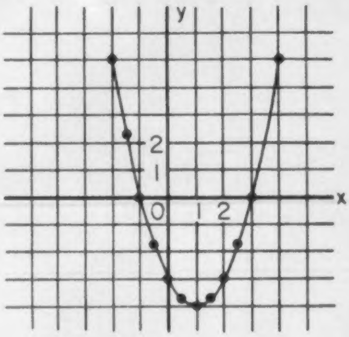


Figure 2.

For all points (x, y) plotted in Figure 2, it is true that $-2 \leq x \leq 4$. Values of x less than -2 and values of x greater than 4 were not listed in the table nor considered in the graph. Keep in mind that the graph of $x^2 - 2x - 3$ has no endpoints when the domain of x is not restricted.

Check Your Reading

1. The graph of the polynomial $Ax^2 + Bx + C$ is the graph of what open sentence?
2. What is the value of the polynomial $x^2 - 2x - 3$ if x is 1?
3. Does the graph of the polynomial $x^2 - 2x - 3$ have endpoints?

Problem Set 18-1b

Draw the graphs of the following polynomials.

1. $2x^2$, for x such that $-2 \le x \le 2$
2. $-2x^2$, for x such that $-2 \le x \le 2$
3. $x^2 - 2$, for x such that $-3 \le x \le 3$
4. $x^2 + x$, for x such that $-3 \le x \le 3$
5. $x^2 + x + 1$, for x such that $-2 \le x \le 3$
6. $-\frac{1}{2}x^2 + x$, for x such that $-3 \le x \le 3$
7. $x^2 - 4x + 4$, for x such that $-1 \le x \le 5$
8. $-x^2 + 4x - 4$, for x such that $-1 \le x \le 5$

From the graph of x^2, which might be called the simplest quadratic polynomial, much can be learned about the graphs of other quadratic polynomials. We begin by comparing the graphs of the following:

$$y = x^2, \quad y = \frac{1}{2}x^2, \quad y = 2x^2, \quad y = -\frac{1}{2}x^2.$$

These graphs will help in understanding the shape of the graph of

$$y = ax^2$$

where a is any non-zero real number.

A list of values of each of the four polynomials (for certain values of x) is given below. Some values have been left for you to determine.

x	-3	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	1	$\frac{4}{3}$	2	3
x^2	9	4		1		0				
$2x^2$	18	8		2		0				
$\frac{1}{2}x^2$	$\frac{9}{2}$	2		$\frac{1}{2}$		0				
$-\frac{1}{2}x^2$	$-\frac{9}{2}$	-2		$-\frac{1}{2}$		0				

The graphs of the four polynomials have been drawn with reference to the same set of axes in Figure 3. Some of the questions in the problem set can best be answered by referring to this figure.

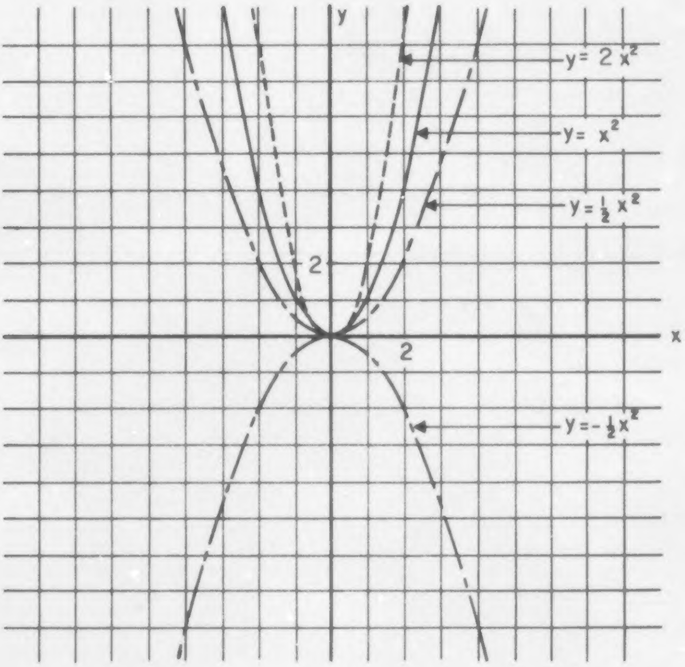


Figure 3.

Check Your Reading

1. What one point is contained in all four graphs? How does this show up in the table of values? in the graph?
2. Notice in the table of values that the values of the four polynomials are the same for 3 as for -3, the same for 2 as for -2, and the same for 1 as for -1. Is there an explanation for this? Give an example of a polynomial for which this would not be true.

Problem Set 18-1c

1. Compare the graphs of x^2 and $2x^2$. Given the graph of x^2, how could the graph of $2x^2$ be drawn without preparing a new table of coordinates? (Hint: Notice that each value of $2x^2$ is twice the corresponding value of x^2.)
2. Draw the graph of x^2 for x such that $-2 \leq x \leq 2$. Then draw the graph of $5x^2$.
3. Compare the graphs of x^2 and $\frac{1}{2}x^2$. Given the graph of x^2, how could the graph of $\frac{1}{2}x^2$ be easily drawn?
4. Draw the graph of x^2 for x such that $-2 \leq x \leq 2$. Then draw the graph of $\frac{1}{4}x^2$.
5. Given the graph of $\frac{1}{2}x^2$, how can the graph of $-\frac{1}{2}x^2$ be obtained from it? Describe how this shows up in the table of values as well as in the graph of Figure 3.
6. Given the graph of x^2, draw the graph of $-x^2$.
7. Explain how the graph of $-ax^2$, where a is any non-zero number, can be obtained from the graph of ax^2.
8. Does it seem correct to say that the shape of the graph of ax^2 is the same as the shape of the graph of x^2? If a is 10, how do the shapes compare? If a is .1, how do the graphs compare?

The previous section showed that the graph of a polynomial ax^2 $(a \neq 0)$ is easily obtained from the graph of x^2. So the problem of graphing all quadratic polynomials of the form ax^2, where a is any non-zero number, has been solved. However, such polynomials form only a subset of all quadratic polynomials. Therefore, the problem of drawing the graph of any quadratic polynomial demands more attention.

Another form of quadratic polynomial that might be considered at this time is illustrated by

$$\frac{1}{2}(x - 3)^2.$$

From the form of this polynomial, it seems possible that its graph may be closely related to the graph of

$$\frac{1}{2}x^2.$$

The graphs of these two polynomials have been drawn in Figure 4.

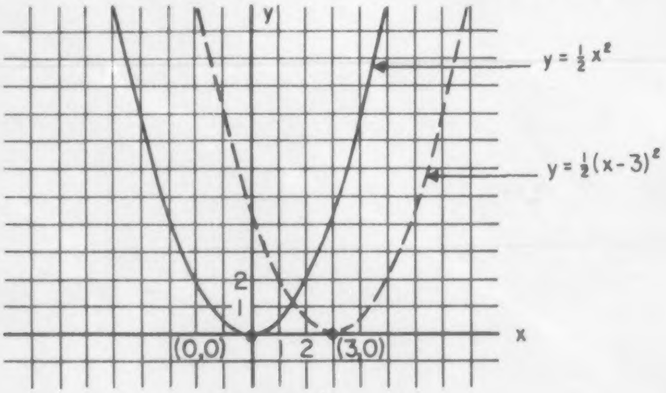


Figure 4

Notice that the graph of "$y = \frac{1}{2}(x - 3)^2$" has the same shape as the graph of "$y = \frac{1}{2}x^2$". The point (3,0) on one graph seems to correspond to the point (0, 0) on the other, and it lies 3 units to the right of (0,0). Similarly, each point of the graph of "$y = \frac{1}{2}(x - 3)^2$" can be thought of as having been obtained by "moving" a corresponding point of the graph of "$y = \frac{1}{2}x^2$" to the right 3 units. More briefly, it might be said that the graph of "$y = \frac{1}{2}(x - 3)^2$" is 3 units to the right of the graph of "$y = \frac{1}{2}x^2$".

> The graph of "$y = \frac{1}{2}(x - 4)^2$" can be obtained by moving the graph of "$y = \frac{1}{2}x^2$" 4 units to the right.
>
> The graph of "$y = \frac{1}{2}(x - 10)^2$" can be obtained by moving the graph of "$y = \frac{1}{2}x^2$" 10 units to the right.
>
> How does the graph of "$y = 5(x - 6)^2$" compare with the graph of "$y = 5x^2$"?

By way of contrast, suppose we had started this section by considering the polynomial $\frac{1}{2}(x + 3)^2$ instead of the polynomial $\frac{1}{2}(x - 3)^2$.

> How do you think the graph of "$y = \frac{1}{2}(x + 3)^2$" compares with the graph of "$y = \frac{1}{2}x^2$"?

You will have a chance to check your answer to this question in the first problem of the problem set.

Check Your Reading

1. In this section, the graph of $\frac{1}{2}(x - 3)^2$ is shown to have the same shape as the graph of what other polynomial?
2. In what way can the graph of $\frac{1}{2}(x - 3)^2$ be obtained from the graph of $\frac{1}{2}x^2$?
3. How does the graph of "$y = 5(x - 6)^2$" compare with the graph of "$y = 5x^2$"?

Problem Set 18-1d

1. From a table of coordinates of points, draw the graphs of

$$y = \frac{1}{2}(x + 3)^2 \quad \text{and} \quad y = \frac{1}{2}x^2$$

with reference to the same set of axes. Does it appear that the graph of "$y = \frac{1}{2}(x + 3)^2$" can be obtained by "moving" the graph of "$y = \frac{1}{2}x^2$"? If so, describe the movement.

2. For each of the following, describe how the graph of the first equation can be obtained from the graph of the second equation.

(a) $y = 3(x + 4)^2$; $\quad y = 3x^2$

(b) $y = 3(x - 4)^2$; $\quad y = 3x^2$

(c) $y = 2(x - 2)^2$; $\quad y = 2x^2$

(d) $y = 2(x + 2)^2$; $\quad y = 2x^2$

(e) $y = -2(x - 3)^2$; $\quad y = -2x^2$

(f) $y = -\frac{1}{2}(x + 1)^2$; $\quad y = -\frac{1}{2}x^2$

(g) $y = \frac{1}{3}(x + \frac{1}{2})^2$; $\quad y = \frac{1}{3}x^2$

*(h) $y = 5(x + 7)^2$; $\quad y = 5(x - 7)^2$

3. If $a \neq 0$, how can the graph of "$y = a(x - h)^2$" be obtained from the graph of "$y = ax^2$", where h is any real number? Be sure to consider the following cases in giving your answer:

$$h > 0, \quad h = 0, \quad h < 0.$$

In the preceding section, it was seen that the graph of

$$y = a(x - h)^2$$

can be obtained by "moving" the points of the graph of "$y = ax^2$" horizontally (to the left or right). It seems natural then to ask if there is a quadratic polynomial whose graph can be obtained by a vertical (up or down) movement of the graph of "$y = ax^2$".

Let us compare the graphs of "$y = \frac{1}{2}x^2$" and "$y = \frac{1}{2}x^2 + 3$". A table of values for each of these polynomials is given below. In Figure 5, the graphs of both have been drawn.

x	-3	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2	3
$\frac{1}{2}x^2$	$\frac{9}{2}$	2	$\frac{1}{2}$	$\frac{1}{8}$	0	$\frac{1}{8}$	$\frac{1}{2}$	2	$\frac{9}{2}$
$\frac{1}{2}x^2 + 3$	$\frac{15}{2}$	5	$\frac{7}{2}$	$\frac{25}{8}$	3	$\frac{25}{8}$	$\frac{7}{2}$	5	$\frac{15}{2}$

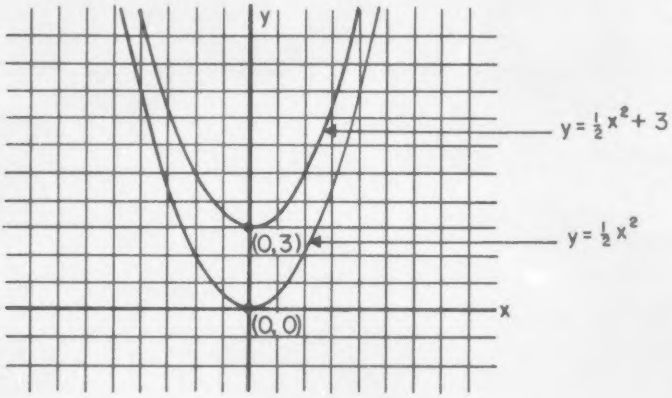


Figure 5

It can be seen both from the table and from the graph that, for any x, the value of $\frac{1}{2}x^2 + 3$ is 3 greater than the value of $\frac{1}{2}x^2$. This means that the graph of $\frac{1}{2}x^2 + 3$ can be obtained by moving the graph of $\frac{1}{2}x^2$ 3 units upward. For example, the point $(0, 3)$ is 3 units above its corresponding point $(0, 0)$. Notice that the shapes of the two graphs are the same.

In Figure 6, the graphs of "$y = \frac{1}{2}(x - 3)^2$" and "$y = \frac{1}{2}(x - 3)^2 - 2$" have been drawn, together with the graph of "$y = \frac{1}{2}x^2$".

Do you see that the graph of "$y = \frac{1}{2}(x - 3)^2 - 2$" has the same shape as the graph of "$y = \frac{1}{2}(x - 3)^2$" but is 2 units below it? For example, the point $(3, -2)$ can be thought of as having been obtained by moving the point $(3, 0)$ 2 units downward.

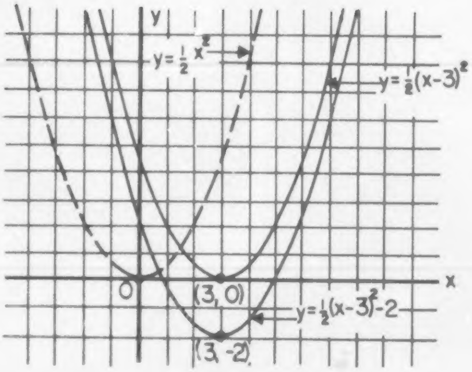


Figure 6

Notice in Figure 6 that the graph of "$y = \frac{1}{2}(x - 3)^2 - 2$" is 3 units to the right of and 2 units below the graph of "$y = \frac{1}{2}x^2$".

> Similarly, the graph of "$y = 2(x + 7)^2 - 5$" is 7 units to the left of and 5 units below the graph of "$y = 2x^2$".
>
> The graph of "$y = -5(x - 3)^2 + 1$" is 3 units to the right of and 1 unit above the graph of "$y = -5x^2$".

At this point, the following summary can be made, based on the graph of

$y = ax^2$, $(a \neq 0)$.

$y = a(x - h)^2 + k$ has a graph which has the same shape as the graph of "$y = ax^2$",

but is $|k|$ units above or below the graph of "$y = ax^2$",

and is $|h|$ units to the right or left of the graph of "$y = ax^2$".

You have probably already made your own observations about how the numbers k and h determine whether the vertical movement is up or down and whether the horizontal movement is to the right or to the left.

The problem of drawing the graph of any quadratic polynomial has now been solved. Every quadratic polynomial, as we shall soon see, can be expressed in the form

$$a(x - h)^2 + k, \qquad a \neq 0,$$

and once this form is determined, the graph is easily obtained from the graph of "$y = ax^2$".

Check Your Reading

1. How can the graph of $\frac{1}{2}x^2 + 3$ be obtained from the graph of $\frac{1}{2}x^2$?
2. How can the graph of "$y = \frac{1}{2}(x - 3)^2$" be obtained from the graph of "$y = \frac{1}{2}x^2$"?
3. In this section, a summary is made of the relationship between the graph of "$y = ax^2$" and the graph of "$y = a(x - h)^2 + k$."
   (a) How do the shapes of the two graphs compare?
   (b) What relationship between the graphs is determined by the number h?
   (c) What relationship between the graphs is determined by the number k?

Oral Exercises 18-1e

1. Which of the following have graphs of the same shape as the graph of "$y = 2(x + 3)^2 - 6$"?
   (a) $y = 2x^2$
   (b) $y = 3(x + 3)^2 - 6$
   (c) $y = 2(x + 3)^2 + 6$
   (d) $y = 2(x - 12)^2 + 157$
   (e) $y = 2(x - 10)^2$
   (f) $y = (x + 3)^2 - 6$
2. What number in the polynomial $a(x - h)^2 + k$ determines the shape of the graph of the polynomial?

Oral Exercises 18-1e

(continued)

3. If $a \neq 0$, describe the way in which the graph of "$y = ax^2$" must be moved (for example, up and to the right) to obtain the graph of "$y = a(x - h)^2 + k$" if:

   (a) $h = 0$ and $k < 0$

   (b) $h = 0$ and $k > 0$

   (c) $h < 0$ and $k = 0$

   (d) $h > 0$ and $k = 0$

   (e) $h < 0$ and $k < 0$

   (f) $h < 0$ and $k > 0$

   (g) $h > 0$ and $k < 0$

   (h) $h > 0$ and $k > 0$

   (i) $h = 0$ and $k = 0$

Problem Set 18-1e

1. Describe how the graph of "$y = x^2 - 2$" and of "$y = x^2 + 2$" can be obtained from the graph of "$y = x^2$". Draw all three graphs with reference to the same set of axes.

2. How can the graph of "$y = 2(x - 2)^2 + 3$" be obtained from the graph of "$y = 2(x - 2)^2$"? Draw both graphs with reference to the same set of axes.

3. How can the graph of "$y = 2(x - 2)^2 + 3$" be obtained from the graph of "$y = 2x^2$"? Draw both graphs with reference to the same set of axes.

4. How is the graph of "$y = (x + 1)^2 - \frac{1}{2}$" obtained from the graph of "$y = x^2$"? Draw both graphs with reference to the same set of axes.

5. How is the graph of "$y = -2(x + \frac{1}{2})^2 + 3$" obtained from the graph of "$y = -2x^2$"? Draw both graphs with reference to the same set of axes.

6. Without drawing the graphs, describe the graph of each of the following by telling how it can be obtained from the graph of some polynomial of the form ax^2.

   (a) $y = 3(x - 7)^2 + \frac{1}{2}$

   (b) $y = 3(x - \frac{1}{2})^2 + 7$

   (c) $y = 2x^2 + \frac{5}{2}$

   (d) $y = 2(x + \frac{5}{2})^2$

   (e) $y = -(x + 3)^2 - 4$

   (f) $y = x^2 + 14$

   (g) $y = 5x^2 + 14$

   (h) $y = 5(x - 2)^2 + 14$

   (i) $y = -8(x - 8)^2 - 8$

   *(j) $y = 4(3 - x)^2 - 6$

Problem Set 18-1e

(continued)

7. Find an open sentence whose graph can be obtained by each of the movements described below.

   (a) the graph of "$y = x^2$" moved 5 units to the right

   (b) the graph of "$y = x^2$" moved 5 units downward

   (c) the graph of "$y = x^2$" moved 5 units upward

   (d) the graph of "$y = x^2$" moved 5 units to the left

   (e) the graph of "$y = x^2$" moved 5 units to the right and 5 units downward

   (f) the graph of "$y = -x^2$" moved $\frac{1}{2}$ unit to the right and 7 units upward

   (g) the graph of "$y = 2x^2$" moved 3 units to the left and 6 units downward

   (h) the graph of "$y = \frac{1}{3}x^2$" moved $\frac{1}{2}$ unit to the right and 1 unit downward

   *(i) the graph of "$y = \frac{1}{2}(x + 7)^2 - 4$" moved 7 units to the right and 4 units upward.

Now that you have learned to draw graphs of quadratic polynomials, it seems wise to introduce several words which allow us to discuss such graphs in a more concise way.

The graph of a quadratic polynomial is called a parabola. You have probably noticed by now that every parabola we have drawn has had either a "highest point " or a "lowest point". Such a point is called the vertex of the parabola. The line that is parallel to the y-axis and passes through the vertex is called the axis of the parabola.

In Figure 7, the graph of "$y = \frac{1}{2}(x - 3)^2 + 2$" has been drawn. The graph is a parabola.

The vertex of the parabola is the point (3, 2).

The axis of the parabola is the line whose equation is "$x = 3$".

Do you see that the vertex of the graph of $a(x - h)^2 + k$, $a \neq 0$, is the point (h, k)? What is its axis?

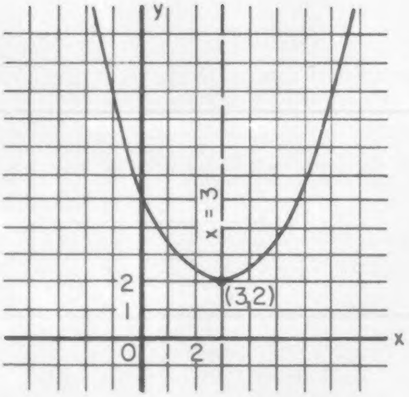


Figure 7.

Check Your Reading

1. What name is given to the graph of any quadratic polynomial?
2. What point of a parabola is called the vertex?
3. What line is called the axis of a parabola?
4. What are the coordinates of the vertex of the graph of "$y = \frac{1}{2}(x - 3)^2 + 2$"?
5. What is the equation of the axis of the graph of "$y = \frac{1}{2}(x - 3)^2 + 2$"?

Oral Exercises 18-1f

Give the coordinates of the vertex and the equation of the axis of each of the following parabolas. (Note that we often speak of an open sentence as a parabola; this means, of course, the parabola which is the graph of the open sentence.)

1. $y = x^2$
2. $y = 5x^2$
3. $y = -5x^2$
4. $y = 5(x - 2)^2$
5. $y = 5(x - 2)^2 + 3$
6. $y = 5(x - 2)^2 - 3$
7. $y = 5(x + 2)^2$
8. $y = 5(x + 2)^2 + \frac{1}{2}$
9. $y = 5(x + 2)^2 - \frac{1}{2}$

*10. $y = a(x - h)^2 + k$

18-2. Standard Forms.

The graph of "$y = (x - 1)^2 - 4$" is a parabola which can be obtained by moving the graph of "$y = x^2$" 1 unit to the right and 4 units downward. This parabola is also the graph of the open sentence

$$y = x^2 - 2x - 3,$$

since it is true that, for any real number x,

$$(x - 1)^2 - 4 = x^2 - 2x - 3. \qquad \text{(Why?)}$$

The above is merely an illustration of a statement that was made at the beginning of this chapter--every quadratic polynomial can be expressed in the form $Ax^2 + Bx + C$. On the other hand, every quadratic polynomial can also be expressed in the form $a(x - h)^2 + k$, and this form is known as the standard form of a quadratic polynomial.

From the example in the first paragraph, it can be seen that changing from standard form to the form $Ax^2 + Bx + C$ is a simple matter. But what about the reverse problem? Given the polynomial $x^2 - 2x - 3$, how could the standard form be determined?

In the standard form, $a(x - h)^2 + k$, the variable x is involved only in an expression that is a perfect square. Therefore, the following approach to the problem might be taken:

Example 1.

$x^2 - 2x - 3 = (x^2 - 2x) - 3$	This is a simple grouping of terms to emphasize that a perfect square is desired in the parentheses.
$= (x^2 - 2x + 1) - 3 - 1$	$x^2 - 2x + 1$ is a perfect square. Note that $1 - 1$ (that is, 0) has been added.
$= (x - 1)^2 - 4.$	We now have the standard form, $a(x-h)^2 + k$, where a is 1, h is 1, and k is -4.

As you can see, the problem of changing to standard form involves the process of completing the square, studied in Chapter 13. Another example is given on the following page. Be sure that you can explain each step.

Example 2.

Change $x^2 + 8x + 2$ to standard form.

$$\begin{aligned} x^2 + 8x + 2 &= (x^2 + 8x) + 2 \\ &= (x^2 + 8x + 16) + 2 - 16 \\ &= (x + 4)^2 - 14. \end{aligned}$$

The method used in examples 1 and 2 is a perfectly general one. It can be used to change any polynomial form $Ax^2 + Bx + C$ to standard form, though not all cases are as simple as the first two examples. A more complicated case is illustrated in Example 3. Be on the lookout for the step in which you seem to be adding 4, but are really adding 12.

Example 3.

Change $3x^2 - 12x + 5$ to standard form.

$3x^2 - 12x + 5 = 3(x^2 - 4x) + 5$	The distributive property has been applied here. You will recall that in the standard form, the coefficient of x^2 in the perfect square is 1. That is the reason for this step.
$= 3(x^2 - 4x + \quad\ ) + 5$	It is merely being emphasized that a perfect square is desired within the parentheses.
$= 3(x^2 - 4x + 4) + 5 - 12$	Here is the step you were warned about! $x^2 - 4x + 4$ is a perfect square. But do you see that 12 (not 4) was added, since multiplication by 3 is distributive over $x^2 - 4x + 4$? This explains the "-12", meaning, in effect, that we have added 12 - 12.
$= 3(x - 2)^2 - 7$	This is the standard form $a(x - h)^2 + k$, where a is 3, h is 2, and k is -7.

Changing to standard form makes drawing the graph a simple matter. On the basis of the three examples above, the following statements can be made:

> The graph of "$y = x^2 - 2x - 3$" is a parabola with vertex $(1, -4)$ and axis "$x = 1$".
>
> The graph of "$y = x^2 + 8x + 2$" is a parabola with vertex $(-4, -14)$ and axis "$x = -4$".
>
> The graph of "$y = 3x^2 - 12x + 5$" is a parabola with vertex $(2, -7)$ and axis "$x = 2$".

Check Your Reading

1. Which of the following is called the standard form of a quadratic polynomial?

   $Ax^2 + Bx + C$; $\qquad a(x - h)^2 + k$

2. Explain why "$(x^2 - 2x + 1) - 3 - 1$" names the same number as "$x^2 - 2x - 3$" for any number x.
3. Explain why "$3(x^2 - 4x + 4) + 5 - 12$" names the same number as "$3(x^2 - 4x) + 5$" for any number x.

Oral Exercises 18-2a

1. Is $(3x - 2)^2 + 5$ in standard form? Why or why not?
2. Describe the first few steps to be taken in changing $3x^2 + 5x + 7$ to standard form.

Problem Set 18-2a

1. The quadratic polynomials below are in standard form. Change each of them to the form $Ax^2 + Bx + C$.

   (a) $(x - 3)^2$ $\qquad$ (d) $-5(x + \frac{1}{2})^2 + 11$

   (b) $2(x - 3)^2$ $\qquad$ (e) $-(x + 3)^2 - 2$

   (c) $2(x - 3)^2 + 6$ $\qquad$ (f) $2x^2 + 5$

Problem Set 18-2a
(continued)

2. The quadratic polynomials below are in the form $Ax^2 + Bx + C$. Change each of them to standard form. (The starred ones, like example 3 in the text, are slightly more complicated.)

(a) $x^2 + 10x - 2$	(g) $x^2 - 3x - 2$
(b) $x^2 + 6x + 10$	(h) $x^2 - 3x + 2$
(c) $x^2 - 15x - 4$	(i) $2x^2 + 5$
(d) $x^2 - 16x$	*(j) $5x^2 - 10x - 5$
(e) $x^2 - 20x + 5$	*(k) $2x^2 + 12x - 7$
(f) $x^2 + x + 1$	*(l) $\frac{1}{2}x^2 - 3x + 2$

3. Change each of the following to the standard form of a quadratic polynomial.

(a) $x^2 - 2x + 5$	(d) $(x + 5)(x - 5)$
(b) $x^2 - x + 2$	*(e) $3x^2 - 2x$
(c) $x^2 + 3x + 1$	*(f) $6x^2 - x - 15$

4. Without drawing the graphs, describe the parabolas which are the graphs of the polynomials in problem 3.

5. Draw the graphs of the following open sentences. In each case, name the points (if any) at which the graph intersects the x-axis.
   (a) $y = x^2 + 6x + 5$
   (b) $y = x^2 + 6x + 9$
   (c) $y = x^2 + 6x + 13$

In Figure 8, the graph of

$$y = x^2 - 3x - 10$$

has been drawn. Can you identify the points at which the graph intersects the x-axis? There seem to be two such points. The value of y at these points is zero. In other words, the abscissas of these points are truth numbers of

$$0 = x^2 - 3x - 10.$$

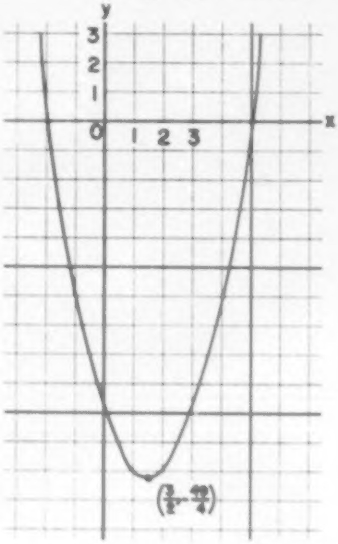
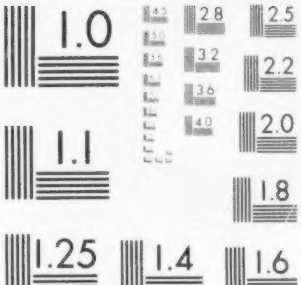


Figure 8

TS


1.0
1.1
1.25
1.4
1.6
1.8
2.0
2.2
2.5
2.8
3.2
3.6
4.0
4.5
5.0

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From the illustration above, it can be seen that there is an important relationship between the graph of "$y = x^2 - 3x - 10$" and the truth set of "$x^2 - 3x - 10 = 0$".

In Figure 9, the graph of

$$y = x^2 - 2x + 2$$

has been drawn. What can be said about the intersection of the graph with the x-axis? Do you see that there are probably no real numbers which make

$$0 = x^2 - 2x + 2$$

true? Thus, the truth set of "$x^2 - 2x + 2 = 0$" should be the empty set.

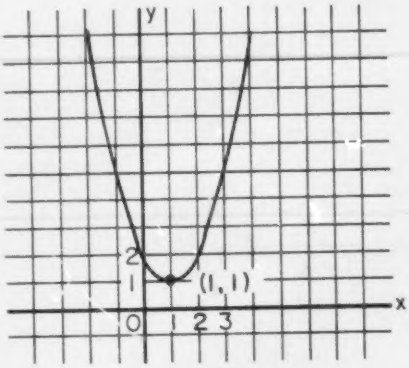


Figure 9

From this illustration it is seen that there is a close relationship between the graph of "$y = x^2 - 2x + 2$" and the truth set of "$x^2 - 2x + 2 = 0$".

In general, how would you describe the relationship between the truth set of the open sentence

$$Ax^2 + Bx + C = 0$$

and the graph of the open sentence

$$y = Ax^2 + Bx + C?$$

Do you agree that the numbers in the truth set of "$Ax^2 + Bx + C = 0$" are the abscissas of the points at which the graph of "$y = Ax^2 + Bx + C$" intersects the x-axis?

Problem Set 18-2b

1. Draw the graph of "$y = x^2 - 4x - 21$".
   Give the abscissa of each point (if any) at which the graph intersects the x-axis.
   What is the truth set of "$x^2 - 4x - 21 = 0$"?

Problem Set 18-2b
(continued)

2. Draw the graph of "$y = x^2 - 10x + 25$".
   Give the abscissa of each point (if any) at which the graph intersects the x-axis.
   What is the truth set of "$x^2 - 10x + 25 = 0$"?

3. Draw the graph of "$y = x^2 + 2$".
   Give the abscissa of each point (if any) at which the graph intersects the x-axis.
   What is the truth set of "$x^2 + 2 = 0$"?

4. How does the graph of "$y = Ax^2 + Bx + C$" show that "$Ax^2 + Bx + C = 0$" probably cannot have more than two real truth numbers?

5. If "$Ax^2 + Bx + C = 0$" has only one real truth number, how would you describe the intersection of the graph of "$y = Ax^2 + Bx + C$" with the x-axis?

6. If "$Ax^2 + Bx + C = 0$" has no real truth numbers, how would you describe the intersection of the graph of "$y = Ax^2 + Bx + C$" with the x-axis?

18-3. Quadratic Equations.

Just as "$Ax^2 + Bx + C$" $(A \neq 0)$ is called a quadratic polynomial, so "$Ax^2 + Bx + C = 0$" is called a quadratic equation. In this section, we shall be concerned with the problem of solving quadratic equations.

The problem of solving quadratic equations is not a new one. In Chapter 13, we solved quadratic equations in which the quadratic polynomial could be factored over the integers. An example of such a case is given below.

The sentences

$$x^2 + 8x + 12 = 0,$$
$$(x + 6)(x + 2) = 0,$$
$$x + 6 = 0 \quad \text{or} \quad x + 2 = 0,$$
$$x = -6 \quad \text{or} \quad x = -2$$

are all equivalent. Hence, the truth set is $\{-6, -2\}$.

Also, we have seen that the truth numbers of a quadratic equation are indicated by the graph of a quadratic polynomial. This graphical solution is, however, an anproximation in most cases. For example, if $\sqrt{2}$ were a truth number, it could not be determined by purely graphical means. (Why?) In this section we shall try to develop an algebraic method for determining the truth set of any quadratic equation.

Example 1. Solve the quadratic equation "$x^2 - 6x - 16 = 0$".

The sentences which follow are all equivalent.

$$x^2 - 6x - 16 = 0$$

$$(x^2 - 6x) - 16 = 0$$

A perfect square is desired in the parentheses.

$$(x^2 - 6x + 9) - 16 - 9 = 0$$

$x^2 - 6x + 9$ is a perfect square. $9 - 9$ (that is, zero) has been added.

$$(x - 3)^2 - 25 = 0$$

$$(x - 3)^2 - (5)^2 = 0$$

This makes it clear that the left member is the difference of two squares.

$$(x - 3 + 5)(x - 3 - 5) = 0$$

For any real numbers a and b $a^2 - b^2 = (a + b)(a - b)$. Here a is $x - 3$; b is 5.

$$(x + 2)(x - 8) = 0$$

$$x + 2 = 0 \quad \text{or} \quad x - 8 = 0$$

$$x = -2 \quad \text{or} \quad x = 8$$

Hence, the truth set is

$$\{-2, 8\}.$$

Thus the truth numbers of "$x^2 - 6x - 16 = 0$" are -2 and 8. This polynomial, $x^2 - 6x - 16$, could have been factored over the integers without completing the square. It serves, however, to illustrate a procedure that can be used to solve any quadratic equation.

Example 2. Solve the quadratic equation "$x^2 - 2x - 2 = 0$".

The sentences which follow are all equivalent.

$$(x^2 - 2x) - 2 = 0$$

$$(x^2 - 2x + 1) - 2 - 1 = 0$$

$x^2 - 2x + 1$ is a perfect square.

$$(x - 1)^2 - 3 = 0$$

Is the left member the difference of two squares?

$$(x - 1)^2 - (\sqrt{3})^2 = 0$$

Since $3 = (\sqrt{3})^2$, we do have the difference of the squares of two real numbers. (Note that we are no longer factoring over the integers.)

$$(x - 1 + \sqrt{3})(x - 1 - \sqrt{3}) = 0$$

$$x - 1 + \sqrt{3} = 0 \quad \text{or} \quad x - 1 - \sqrt{3} = 0$$

$$x = 1 - \sqrt{3} \quad \text{or} \quad x = 1 + \sqrt{3}$$

Hence, the truth set is

$$\{1 - \sqrt{3},\ 1 + \sqrt{3}\}.$$

Example 3. Solve the quadratic equation "$x^2 - 2x + 2 = 0$".

$$(x^2 - 2x) + 2 = 0$$

$$(x^2 - 2x + 1) + 2 - 1 = 0$$

$$(x - 1)^2 + 1 = 0$$

This sentence is equivalent to "$x^2 - 2x + 2 = 0$" and its truth set is $\emptyset$.

In this step, it can be seen that no real number will make this sentence true. $(x - 1)^2$ can never be less than zero. (Why?) So $(x - 1)^2 + 1$ can never be less than one. Compare this with the discussion of this same equation in section 18-2b.

In the next example, the coefficient of x^2 in the quadratic equation is not one. This makes some of the steps slightly more complicated, but the basic method does not change.

Example 4. Solve the quadratic equation "$2x^2 - 5x - 12 = 0$".

The sentences which follow are all equivalent.

$$2x^2 - 5x - 12 = 0$$

$$2(x^2 - \frac{5}{2}x) - 12 = 0$$

The distributive property has been used here so that the coefficient of x^2 within the parentheses is one.

$$2(x^2 - \frac{5}{2}x + \frac{25}{16}) - 12 - \frac{50}{16} = 0$$

$x^2 - \frac{5}{2}x + \frac{25}{16}$ is a perfect square.

$\frac{50}{16} - \frac{50}{16}$ (that is, zero) has been added.

$$2(x - \frac{5}{4})^2 - \frac{242}{16} = 0$$

$- \frac{242}{16}$ is a simpler name for $- 12 - \frac{50}{16}$.

$$(x - \frac{5}{4})^2 - \frac{121}{16} = 0$$

Both members have been multiplied by $\frac{1}{2}$, giving an equivalent sentence. This makes it easy to identify $(x - \frac{5}{4})^2$ as a perfect square.

$$(x - \frac{5}{4})^2 - (\frac{11}{4})^2 = 0$$

Once again, the left member is the difference of two squares.

$$(x - \frac{5}{4} + \frac{11}{4})(x - \frac{5}{4} - \frac{11}{4}) = 0$$

$$(x + \frac{6}{4})(x - \frac{16}{4}) = 0$$

$$x + \frac{6}{4} = 0 \quad \text{or} \quad x - \frac{16}{4} = 0$$

$$x = - \frac{3}{2} \quad \text{or} \quad x = 4$$

Hence, the truth set is

$\{- \frac{3}{2}, 4\}$.

Problem Set 18-3a

1. Factor the following expressions over the real numbers using the method of the difference of two squares.

   (a) $(x - 1)^2 - 4$

   (b) $(x + 2)^2 - 9$

   (c) $(x - 2)^2 - 1$

   (d) $(x + \frac{3}{2})^2 - 1$

   (e) $(x - \frac{7}{2})^2 - \frac{25}{4}$

   (f) $(x - 1)^2 - (\sqrt{3})^2$

   (g) $(x + 3)^2 - 5$

   (h) $(x - \frac{1}{2})^2 - 7$

   (i) $(x - \frac{3}{2})^2 - \frac{3}{4}$

   (j) $(x + \frac{5}{6})^2 - \frac{5}{36}$

2. Solve the following quadratic equations.

   (a) $x^2 + 2x - 3 = 0$

   (b) $x^2 + 4x + 2 = 0$

   (c) $x^2 + 6x + 3 = 0$

   (d) $x^2 + 6x - 3 = 0$

   (e) $x^2 + 10x + 6 = 0$

   (f) $x^2 + 6x + 4 = 0$

   (g) $x^2 + 4x + 6 = 0$

   (h) $x^2 - 6 = 0$

   (i) $x^2 + 6 = 0$

   (j) $2x^2 = 4x - 11$

   (k) $2x^2 - 5x = 12$

   (l) $2x^2 - 7x + 3 = 0$

Problem Set 18-3a
(Continued)

3. The length of a rectangle is 6 feet more than the width. The area of the rectangle is 36 square feet. Find the length and the width of the rectangle.

4. One number is 5 less than another number. The product of the numbers is 100. Find the numbers.

5. The sum of a number and its reciprocal is 4. Find the number.

6. The sum of a number and its reciprocal is -4. Find the number.

7. The sum of 14 times a number and the square of the number is 11. Find the number.

8. The length of a rectangular piece of sheet metal is 3 feet more than the width. If the area is $46\frac{3}{4}$ square feet, find the length.

*9. The sum of a number and its reciprocal is 0. Find the number.

*10. An open box is constructed from a rectangular sheet of metal 8 inches longer than it is wide as follows:

> Out of each corner a square of side 2 inches is cut, and the sides are folded up. The volume of the resulting box is 256 cubic inches.

What were the dimensions of the original sheet of metal?

*11. A rope hangs from the window of a building. If pulled taut vertically to the base of the building there is 8 feet of rope lying slack on the ground. If pulled out taut until the end of the rope just reaches the ground, it reaches the ground at a point which is 28 feet from the building. How high above the ground is the window?

*12. John drove 336 miles to Chicago, bought a new car and returned the next day by the same route. The return trip took 1 hour longer than the original trip, and his average speed was 6 miles per hour slower. Find his average speed each way.

Summary

1. A polynomial of form $Ax^2 + Bx + C$, where A, B, and C are any real numbers, with $A \neq 0$, is called a *quadratic polynomial* in x.

2. Every quadratic polynomial in x may be expressed in the *standard form* $a(x - h)^2 + k$.

3. The graph of a quadratic polynomial $Ax^2 + Bx + C$ is the set of points (x,y) satisfying the open sentence "$y = Ax^2 + Bx + C$" and is called a *parabola*.

4. The graph of "$y = a(x - h)^2 + k$" has the same shape as the graph of "$y = ax^2$" and may be obtained by moving the graph of "$y = ax^2$" horizontally and vertically according to the values of h and k.

5. An open sentence of the form, "$Ax^2 + Bx + C = 0$", where A, B, and C are real numbers, with $A \neq 0$, is called a *quadratic equation*.

6. The truth numbers of "$Ax^2 + Bx + C = 0$" are the abscissas of the points at which the graph of "$y = Ax^2 + Bx + C$" intersects the x-axis.

7. A quadratic equation in x, "$Ax^2 + Bx + C = 0$" has either two real truth numbers, one real truth number, or no real truth numbers.

Review Problem Set

1. Describe how the graph of each of the following open sentences can be obtained from the graph of $y = \frac{1}{3}x^2$.

(a) $y = \frac{1}{3}x^2 - 2$	(e) $y = \frac{1}{3}(x - 6)^2 + 8$
(b) $y = \frac{1}{3}(x - 2)^2$	(f) $y = \frac{1}{3}(x + 5)^2 + 10$
(c) $y = \frac{1}{3}(x + 7)^2$	(g) $y = \frac{1}{3}(x + \frac{1}{4})^2 - 2.5$
(d) $y = \frac{1}{3}x^2 + \frac{1}{2}$	(h) $y = \frac{1}{3}(x - n)^2 + t$; $(n > 0,\ t > 0)$

Review Problem Set
(continued)

2. Write each of the following quadratic polynomials in the standard form $a(x - h)^2 + k$. In each case, give the values of a, h, and k.

   (a) $x^2 + 6x + 2$
   (b) $x^2 - 10x - 7$
   (c) $x^2 - 4x + 4$
   (d) $x^2 + 12x + 20$
   (e) $x^2 - 7x - 2$
   (f) $x^2 + 5x + 40$
   (g) $2x^2 + 8x + 7$
   (h) $2x^2 - 9x + 12$

3. Write each of the following quadratic polynomials in standard form. Draw the graph of the polynomial from 3 units to the left of the vertex to 3 units to the right of the vertex.

   (a) $x^2 + x - 6$
   (b) $x^2 + 12x + 32$
   (c) $x^2 - 6x + 9$
   (d) $x^2 + x + 1$

4. Use the graphs of problem 3 to determine the truth numbers of the following quadratic equations.

   (a) $x^2 + x - 6 = 0$
   (b) $x^2 + 12x + 32 = 0$
   (c) $x^2 - 6x + 9 = 0$
   (d) $x^2 + x + 1 = 0$

5. Determine the truth set of each of the following open sentences.

   (a) $x^2 + 8 = 6x$
   (b) $x^2 + 21 = 10x$
   (c) $x^2 = 2x + 1$
   (d) $x + 6x^2 = 1$
   (e) $x^2 - 4x + 1 = 0$
   (f) $x^2 + 7x + 6 = 0$
   (g) $x^2 + x + 5 = 0$
   (h) $x^2 + 2x + 2 > 0$

6. Give the coordinates of the vertex and the equation of the axis of each of the following parabolas.

   (a) $y = 3x^2 + 2$
   (b) $y = x^2 - 7$
   (c) $y = (x - 4)^2$
   (d) $y = 2(x + 7)^2$
   (e) $y = 2(x - 7)^2 - \frac{1}{2}$
   (f) $y = -4(x + 2)^2 - 8$
   (g) $y = -3(x - 2)^2 - 4$
   (h) $y = 6(x + 5)^2 + 10$

Review Problem Set
(continued)

7. In each of the following tell whether the graph is a point, a line, a pair of lines, a parabola, or none of these.

   (a) $y = x$

   (b) $y = x^2$

   (c) $y = x + 7$

   (d) $y = x^2 + 7$

   (e) $y^2 - x^2 = 0$ (Hint: factor the polynomial.)

   (f) $y = (x + 7)^2$

   (g) $x - y = 4$

   (h) $x^2 - y = 4$

   (i) $(x - 3)^2 = 0$

   (j) $x = 0$

   (k) $y = 0 \cdot x^2$

   (l) $x - 3 = y$ and $2x - y = 3$

   (m) $x - 3 = y$ or $2x - y = 3$

8. One number is 7 more than another number. The square of the larger number is twice the square of the smaller number. Find the numbers.

9. One number is 7 more than another number. The square of the smaller number is twice the square of the larger number. Find the numbers.

10. The perimeter of a rectangle is 94 feet, and its area is 496 square feet. What are its dimensions?

*11. The sum of two numbers is 9. Find the numbers and their product if the product is the largest number possible.

*12. A boat manufacturer finds that his cost per boat in dollars is related to the number of boats manufactured each day by the formula

$$c = n^2 - 12n + 175.$$

Find the number of boats he should manufacture each day so that his cost per boat is smallest.

Chapter 19

FUNCTIONS

19-1. The Function Idea.

In working with sets in Chapter 1 we were given such problems as this. Given a set

$$D = \{1, 2, 3, 4, 5\},$$

form a new set by multiplying each number in the set D by 7. You would write the new set as

$$R = \{7, 14, 21, 28, 35\}.$$

Looking over these two sets we see that with the number 1 in D we associate the number 7 in R, with the number 2 in D we associate the number 14 in R, and so forth. Do you see that with every number in D we associate some number in R?

In other words, if a number in D were called out, and you were asked to give the corresponding number in R, you could always do this.

Let's take another example. Suppose we think of a set D as consisting of all the positive even integers, that is,

$$D = \{2, 4, 6, 8, \ldots\}.$$

Now form a set R by adding $5\frac{1}{2}$ to each element in D. It is easy to see what the elements in R would be. For R you would write

$$R = \{7\tfrac{1}{2}, 9\tfrac{1}{2}, 11\tfrac{1}{2}, 13\tfrac{1}{2}, \ldots\}.$$

Once again we see that with every element in D we associate some element in R. If someone gave you the number 20 in D and asked for the corresponding element in R, you would undoubtedly say $25\frac{1}{2}$.

It is possible to make the association because of a rule. The rule in this case was addition of $5\frac{1}{2}$. In the first case the rule was multiplication by 7.

Suppose you are working on a traffic survey. You are asked to keep a record of the number of cars which cross a certain bridge each day. Your job is to do this every day for ten days. At the end of the ten days you turn in your report. It looks like this

Day	1	2	3	4	5	6	7	8	9	10
Number of cars	100	160	98	80	200	80	156	140	150	150

Though it may not seem so at first, this record of yours represents a situation very similar to the two cases on the previous page. We can think of the top row, representing the first, second, third day, etc., as the set of integers from 1 to 10. The bottom row is also a set of numbers. How many elements does this second set have? There are two 80's and two 150's. We count each number once. Therefore the second set has 8 elements.

You don't have a specific rule telling you to do something to the numbers in the top set in order to get corresponding numbers in the second set. However, the chart itself acts as a rule. It gives you with each day a corresponding number, the number of cars for that particular day. Thus if you were asked, for example, to give the number which is associated with day number 7, you could immediately say 156. You could also give a specific answer for each of the other days.

As another example, suppose we consider the following set of temperatures given in degrees Fahrenheit.

$$D = \{212, \ 50, \ 32, \ 21, \ -4, \ -40\}$$

If we are asked to change each of these to degrees Centigrade, we would use the formula

$$C = \frac{5}{9}(F - 32).$$

For example, if $F = 212$, then $C = \frac{5}{9}(212 - 32)$

$$= 100.$$

If we apply this formula to all temperatures in the set D, we obtain a new set of temperatures

$$R = \{100, \ 10, \ 0, \ -5, \ -20, \ -40\}.$$

Check these values!

In this case the formula $C = \frac{5}{9}(F - 32)$ associates with each Fahrenheit temperature in D a corresponding Centigrade temperature in R. With what element in D do we associate the number 0 in R?

A final example illustrating the same general idea is the following. Consider the set D to be the set of all real numbers. Consider also the open phrase

$$x^2 - 3.$$

If we let x have a value from the set D, for instance 10, then the number represented by the phrase $x^2 - 3$ is 97. We call 97 the element in the second set R which is associated with the element 10 in D. We could say that 97 is the number assigned to the number 10 by the rule. The phrase $x^2 - 3$ describes the rule.

What, in this case, is the set R? Do you see that it is the set of all real numbers greater than or equal to -3? Why?

Check Your Reading

1. The set $D = \{1, 2, 3, 4, 5\}$ and the rule "multiply by seven" determine what set?
2. "Multiplying by seven" and "adding $5\frac{1}{2}$" are two rules used in this section. What are two other rules used in this section?
3. If D is the set of real numbers, then what number does the phrase $x^2 - 3$ assign to 10? What number in R is associated with 5 in D?

Oral Exercises 19-1a

Given the set $D = \{1, 2, 3\}$. Find the set R of numbers obtained by

1. adding 6 to each element of D.
2. multiplying each element of D by 3.
3. squaring each element of D.
4. squaring each element of D and adding 1 to the square.
5. using the rule described by the phrase $x^2 + 3$ where x is an element of D.

Oral Exercises 19-1a
(continued)

6. assigning to each element of D the sum of the element and its successor.
7. assigning to each even number in D the number zero, and to each odd number in D its square.
8. assigning to each number in D, the number 7.
9. associating with each element in D, its successor.
10. associating with each element in D, its cube.

Problem Set 19-1a

1. Given the set $D = \{1, 2, 3\}$. Find the set R of numbers obtained by
   (a) taking the opposite of each element of D.
   (b) using the rule described by the phrase $2m$ where m is an element of D.
   (c) using the rule described by the phrase $|x|$ where x is an element of D.
   (d) using the rule described by the phrase $7 - x$ where x is an element of D.
   (e) using the rule described by the phrase $\frac{1}{y}$ where y is an element of D.
   (f) using the rule described by the phrase $3z - 4$ where z is an element of D.

2. Find the set R given the following information.
   (a) D is the set of whole numbers and the rule is "to each element of D add one".
   (b) D is the set of whole numbers and the rule is "to each element in D assign its successor".
   (c) D is the set of whole numbers and R is obtained by using the rule described by the phrase $x + 1$ where x is an element of D.

Problem Set 19-1a
(continued)

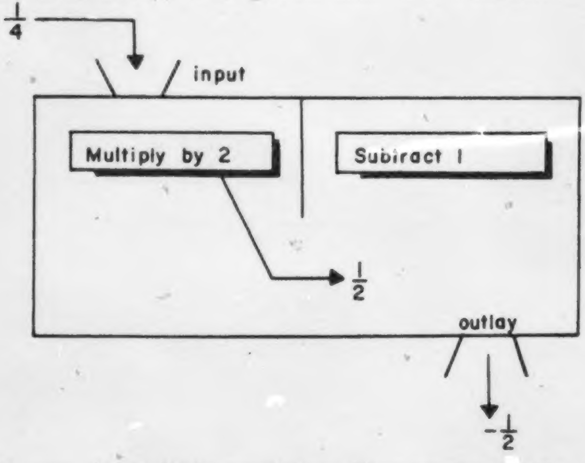
3. Find the set R given the following information.
   (a) D is the set of real numbers; to each element of D assign the number 7.
   (b) D is the set of real numbers; to each rational number in D assign the number 1 and to each irrational number in D assign the number -1.
   (c) D is the set of real numbers; to each negative number in D assign the number -1, to 0 in D assign the number 0, and to each positive number in D assign the number 1.

4. In each of the following, describe carefully the sets D and R of the function.
   (a)

D	positive integer n	1	2	3	4	5	6	7	8	9	10	...
R	n^{th} odd integer	1	3	5	7	9						

Fill in the missing numbers. What number in R is associated with 13 in D? With 1000 in D? What is the rule?

   (b) Imagine a special computing machine which accepts any positive real number, multiplies it by 2, subtracts 1 from this, and gives out the result.



If you feed this machine the number 17, what will come out? What number does the machine associate with 0? With -1?

Problem Set 19-1a
(continued)

(c) Draw two parallel real number lines and let the unit of measure on the upper line be twice that on the lower line. Place the lower line so that its point 1 is directly below the point 1 on the upper line. Now for each point on the upper (first) line there is a point directly below on the second line. What number is below -13? Below 13? What number on the second line is associated with 1000 on the first line by this arrangement?

(d) Draw a line with respect to a set of coordinate axes such that its slope is 2 and the y-coordinate of its y-intercept is -1. For each number a on the x-axis there is a number b on the y-axis such that (a, b) are the coordinates of a point on the given line. If we pick -1 on the x-axis, the line associates with -1 what number on the y-axis? What number on the y-axis is associated with $-\frac{1}{2}$ on the x-axis? With 5? With -5? With 13?

(e) For each real number t such that $|t| < 1$, use the rule described by the expression "2t - 1" to obtain an associated number. What number does this expression associate with $\frac{1}{2}$? With $-\frac{1}{2}$? With $\frac{2}{3}$? With $-\frac{2}{3}$? With 2?

(f) Given any negative real number, multiply it by 2 and then subtract 1. What number does this verbal instruction associate with -8? With -13? With 0?

In all the examples and problems we have been working with in this chapter the central idea has been the same. We have been given a set of numbers and a rule. The rule assigns to each number in this set exactly one number. The assigned numbers form a second set. To bring out the importance of the idea we have special names. The basic idea is described as follows:

> Given a set of numbers and a rule which assigns to each number of this set exactly one number, the resulting association of numbers is called a
>
> function.
>
> The given set is called the domain of definition of the function, and the set of assigned numbers is called the range of the function.

In most of our examples we have called the domain of definition (when there is no confusion we use the single word domain) by the letter D. For the range we have used the letter R. The word function really applies to all three parts, that is, the domain, the rule, and the range. It is true, however, that the range is completely determined if we know the rule and the domain.

Notice that we have said that our rule assigns to each element in D exactly one number. It is possible to have a rule which assigns more than one number to each element in a given set. For example we might call the set D the set of squares of integers beginning with 1. In other words, let

$$D = \{1, 4, 9, 16, 25, \ldots\}.$$

Then we might have a rule which assigns to each element in D its square roots. Thus with the number 9 in D we would associate the numbers 3 and -3, etc. In a case like this even though we have a domain, a rule, and a range, we do not call such an association a function.

Sometimes the domain of definition of a function is not stated explicitly. It is then understood to be the largest set of real numbers to which the rule for the function can be sensibly applied. For example, if a function is described by the expression $\frac{1}{x+2}$, then, unless stated otherwise, its domain of definition is understood to be the set of all real numbers different from -2. Why? Similarly the domain of definition of the function defined by $\sqrt{x+2}$ is understood to be the set of all real numbers greater than or equal to -2. Why?

It is possible to define a function by means of a set of ordered number pairs, in which the second number of each pair is the number assigned to the first. In other words, the domain is the set of first numbers, while the range is the set of second numbers.

For example the first function given in the chapter could have been written as

$$\{(1, 7), (2, 14), (3, 21), (4, 28), (5, 35)\}.$$

The function defined by the table on page 844 could have been written as

$$\left\{\begin{array}{l}(1, 100), (2, 160), (3, 98), (4, 80), (5, 200), (6, 80), \\ \qquad (7, 156), (8, 140), (9, 150), (10, 150).\end{array}\right\}$$

On the other hand, the following set of ordered pairs does not define a function.

$$\{(2, 5), (3, 7), (4, 8), (2, 6)\}$$

Do you see why it does not define a function? Remember that a function assigns to every number in the domain exactly one number in the range.

Check Your Reading

1. What names are given the two sets of a function?
2. How many elements of the range may be assigned to each element of the domain of a function?

Check Your Reading
(continued)

3. What sort of an association is described in the text which has a domain, a rule, and a range, but is not called a function?

Oral Exercises 19-1b

Describe the domain and range of the following functions.

1. To each number in the set $\{1, 2, 3\}$ assign five times the number.
2. To each number in the set $\{2, 3, 5\}$ assign its square.
3. Associate the positive square root with each element of the set $\{4, 9, 16, 25\}$.
4.

Time A.M.	8	9	10	11	12
Temperature	62	65	70	71	71

5. To each real number assign its opposite.
6. $2n$ is associated with n, an integer such that
$$0 < n < 5.$$
7. $3x$ is associated with x, where x is an element of the set $\{0, 1, 2\}$.
8. $4x - 1$ is associated with x, where x is a real number in the set $\{2, \sqrt{2}, \sqrt[3]{2}\}$.
9. To each real number x, assign a number y such that (x, y) is a solution of the equation
$$y = 3x + 1.$$
10. To each two-digit positive integer assign the sum of its digits.

Problem Set 19-1b

1. Describe the domain and range of the following functions.
   (a) With each positive integer associate its remainder after division by 5.
   (b) To each positive real number assign the product of $\frac{1}{3}$ and two more than the number.
   (c) To each integer, n, in the set $\{1, 2, 3, \ldots, 12\}$ assign the number of days in the n^{th} month of the (non-leap) year.
   (d) With each real number associate twice its opposite.
   (e) With each fraction in the set $\{\frac{1}{4}, \frac{1}{8}, \frac{3}{4}, \frac{1}{16}, \frac{1234}{10000}\}$ associate the number of decimal places in its decimal representation.
   (f) With each number n in the set $\{1, 2, 3, \ldots, 10\}$ associate the n^{th} prime number, for example with the number 3 associate the 3rd prime number, etc.
   (g) To each even number in the set $A = \{1, 2, 3, \ldots, 12\}$ assign the number 0 and to each odd number in A assign the number 1.

2. In each of the following describe the functions in two ways: (i) by a table, (ii) by an expression in x. In each case describe the domain of definition and the range.
   (a) To each positive integer assign the product of 3 and the number.
   (b) To each positive integer less than 10 assign the sum of 5 and the number.
   (c) With each positive integer less than 8 associate the sum of 3 times the number and 2.
   (d) With each positive integer associate its successor.
   (e) With each positive integer associate its opposite.
   (f) To each number which is a square of a positive integer assign its positive square root.

3. With each positive integer greater than 1 associate the smallest factor of the integer (greater than 1). Form a table for the first ten positive integers (greater than 1). What integers are associated with themselves?

Problem Set 19-1b
(continued)

4. The cost of mailing a package is determined by the weight of the package to the greatest pound. This can be described as: To every positive real number (weight in pounds) assign the smallest integer which is greater than or equal to it. What integers would be assigned to each of the following:

$$\frac{3}{4},\quad 1\frac{1}{2},\quad 1\frac{5}{8},\quad 2,\quad 2\frac{3}{8}\ ?$$

Is this association a function?

5. Find the domains of definition of the functions defined by the following expressions:

(a) $\frac{x}{x - 3}$

(b) $\sqrt{2x - 2}$

(c) $3 - \frac{1}{x}$

(d) $\sqrt{x^2}$

(e) $\sqrt{x^2 - 1}$

(f) $\frac{3}{x^2 - 4}$

6. Which of the following sets of number pairs define functions?

(a) $(-1, 2), (0, 3), (1, 4), (2, -1)$

(b) $(1, 2), (2, 3), (3, 2), (4, 3)$

(c) $(1, 2), (1, 3), (2, 4), (2, 5)$

(d) $(2, 2), (3, 2), (4, 2), (5, 2)$

(e) $(3, 1), (3, 2), (3, 3), (3, 4)$

(f) $(\frac{1}{2}, -1), (-\frac{1}{2}, 1), (\frac{3}{2}, -1), (-\frac{3}{2}, 1)$

19-2. The Function Notation.

We have been using letters as names of numbers. We shall now use letters such as f, g, h, J, etc., as names for functions. If f is the name of a given function and x is a number in its domain, then we shall use the symbol $f(x)$ to represent the number which the function f assigns to x. The number represented by $f(x)$ is called the value of f at x. Thus $f(x)$ represents a number in the range. Notice that $f(x)$ does not mean "f times x".

For example, when we wish to describe the function f as that which assigns to each real number x the real number $2x - 1$, we may write

$$f(x) = 2x - 1, \text{ for each real number } x.$$

Then

$$\begin{aligned} f(5) &= 2(5) - 1 \\ &= 9 \end{aligned}$$

and
$$\begin{aligned} f(\tfrac{1}{4}) &= 2(\tfrac{1}{4}) - 1 \\ &= -\tfrac{1}{2}. \end{aligned}$$

Similarly
$$\begin{aligned} f(0) &= 2(0) - 1 \\ &= -1. \end{aligned}$$

Also $f(a) = 2a - 1$, for any real number a.

Check to see if you agree with the following.

$$f(10) = 19, \quad f(\tfrac{4}{3}) = \tfrac{5}{3}, \quad f(-7) = -15, \quad f(s) = 2s - 1,$$

where s is a real number. Suppose t is a real number. For the same function f what is $f(2t)$? Do you see that

$$\begin{aligned} f(2t) &= 2(2t) - 1 \\ &= 4t - 1 \end{aligned}$$

and
$$\begin{aligned} f(t - 1) &= 2(t - 1) - 1 \\ &= 2t - 2 - 1 \\ &= 2t - 3 \text{ ?} \end{aligned}$$

Sometimes a function is defined in two or more parts. We could define, for example, a function j as follows:

$$\begin{cases} j(x) = x^2, \text{ for each number } x \text{ such that } x \geq 0; \\ j(x) = 5x, \text{ for each number } x \text{ such that } x < 0. \end{cases}$$

This is really a single rule even though it is described by means of two equations. The domain of the function is the set of all real numbers. However, for a non-negative number in the domain our rule tells us to square the number. Thus

$$j(4) = 16.$$

But for a negative number in the domain the rule tells us to multiply by 5. So

$$j(-4) = -20.$$

Do you see that

$$j(10) = 100, \quad j(-10) = -50, \quad j(0) = 0 \ ?$$

There is a shorter way of writing a description of a function such as j. We can write it as follows:

$$j(x) = \begin{cases} x^2, & x \geq 0 \\ 5x, & x < 0 \end{cases}$$

The following function h is also of this type. It is defined by

$$\begin{cases} h(x) = x, & \text{for each number } x \text{ such that } x \geq 0; \\ h(x) = -x, & \text{for each number } x \text{ such that } x < 0. \end{cases}$$

Again it is a single rule, and it defines one function, even though it involves two equations. Once more we write this type of rule in the shorter form

$$h(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Let us examine this function carefully to see just what it means. We can do this by taking various elements in the domain, and then seeing what the assigned numbers are. Take, for example, the number 5. Since this is greater than zero, we use the top line. In this case we see that

$h(5) = 5.$ Do you see why?

But now suppose we take the number -5. In this case we have a number less than zero, so we use the bottom line. On the bottom line we are told that the function h assigns to a negative number its opposite. Thus

$$h(-5) = 5.$$

Likewise $h(-12) = 12,$

while $h(12) = 12.$

Do you also see that $h(0) = 0$?

We have worked with this function before, only we gave it another name. Do you see that it is the same as the function defined by

$$h(x) = |x|, \text{ for every real number } x?$$

We say that two functions are the same if they involve the same domain and determine the same association of numbers. Otherwise they are different. In the case of the following two functions

$$f(x) = 7x, \text{ for all integers } x,$$

and

$$q(x) = 7x, \text{ for all real numbers } x,$$

the two rules appear to be the same, but the domains are different. Thus f and q are not the same function. In this connection we should realize that a function has no value at a number outside its domain. For example if a function is defined as $f(x) = x^2$, for all integers x, then $f(\frac{1}{2})$ has no meaning, even though we might be tempted to say $f(\frac{1}{2}) = \frac{1}{4}$.

Let us consider another function g defined by the rule

$$\begin{cases} g(x) = -1, \text{ for each real number } x \text{ such that } x < 0, \\ g(x) = 0, \text{ for } x = 0, \\ g(x) = 1, \text{ for each real number } x \text{ such that } x > 0. \end{cases}$$

It is important to understand that this is also a single rule. It is described in three parts. We again write the short form

$$g(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

Do you see that the domain is the set of all real numbers? What is the range? It should be clear that $R = \{-1, 0, 1\}$. We can get a better understanding of g by checking the following:

$g(5) = 1$, $g(-5) = -1$, $g(0) = 0$, $g(3.92) = 1$,
$g(\sqrt{3}) = 1$, $g(\sqrt[3]{-8}) = -1$.

If a is any number greater than zero, what is $g(a)$?

If a is any non-zero real number, what is $g(|a|)$?

Check Your Reading

1. What number does the function f, in the example at the beginning of this section, assign to x, a number in the domain of definition?

2. If to each real number x the function f assigns the real number $2x - 1$, what real numbers are represented by $f(5)$, $f(0)$, $f(3)$, $f(\frac{1}{2})$?

3. In the text $h(x)$ is defined in two ways. What are they?

4. What are the domain and range of the function g as defined in the text? If $a > 0$, what is $g(a)$?

5. If $f(x) = 7x$ and $g(x) = 7x$, under what conditions are f and g the same function? Under what conditions are they different functions?

Oral Exercises 19-2

1. If "$g(x) = 2x$" defines a function, what number is represented by $g(3)$, $g(-2)$, $g(0)$, $g(\frac{1}{2})$, $g(a)$, $g(2a)$?

2. If "$F(x) = 3$" defines a function, what number is represented by $F(3)$, $F(-2)$, $F(0)$, $F(a)$?

3. If H is a function defined by

$$H(x) = \begin{cases} \sqrt{x}, & \text{for } x \geq 0 \\ 0, & \text{for } x < 0, \end{cases}$$

find the number represented by each of the following: $H(-\frac{1}{2})$, $H(9)$, $H(\frac{16}{25})$, $H(-25)$.

4. Let f be a function defined by "$f(x) = 2x$" for all integers x and let g be a function defined by "$g(x) = 2x$" for all rational numbers x. Are f and g the same function? Why?

5. Given: f is a function defined by "$f(x) = 2x$" for all integers x, and g is a function defined by "$g(t) = 2t$" for all integers t. Are f and g the same function?

Problem Set 19-2

1. If f is a function defined by "$f(x) = x + 1$", find the numbers represented by $f(1)$, $f(\frac{1}{2})$, $f(-\frac{3}{2})$, $f(0)$, $f(a)$. What is the domain of f?

2. If F is a function defined by "$F(t) = 2 - \frac{t}{2}$" for each real number t, find the numbers represented by:

   (a) $F(-2)$

   (b) $-F(2)$

   (c) $F(-\frac{1}{2})$

   (d) $F(1) - 1$

   (e) $F(0)$

   (f) $|F(-6)|$

   (g) $F(|-6|)$

   (h) $F(a)$, for any real number a

   (i) $F(\frac{a}{2})$

   (j) $F(2a)$

   (k) $F(\frac{1}{a})$, $a \neq 0$

3. Given the function G defined by: $G(t) = |t|$ for each real number t,

   (a) what is the domain of G?

   (b) what is the range of G?

   Find the real numbers represented by:

   (c) $G(3)$

   (d) $G(-3)$

   (e) $G(0)$

   (f) $G(4) - G(4)$

   (g) $G(4) - G(-4)$

   (h) $\frac{G(-3)}{3}$

   (i) $5G(5)$

4. Given: g is a function described by the rule "to each positive integer x assign its successor".

   (a) Write an expression in x for $g(x)$.

   (b) What is the domain of g?

   (c) What is $g(-1)$, $g(\frac{1}{2})$?

   (d) Is g the same function as f of problem 1? Why?

Problem Set 19-2
(continued)

5. Given: h is a function defined by

$$h(t) = \begin{cases} -1, & t < 0 \\ 0, & t = 0 \\ 1, & t > 0 \end{cases}$$

How does this function differ from the function g defined in Section 19-2? (Notice the same function may have different names and involve different letters representing the variables, provided the rule and the domain remain the same.)

6. Let F be the function defined in problem 2. What is the truth set of each of the following sentences?

(a) $F(x) = -1$

(b) $F(x) < 0$

(c) $F(x) = -\frac{1}{2}$

(d) $F(x) = x$,

(e) $F(x) > 2$

(f) $F(x) \leq 1$

7. The following set of number pairs defines a function f. (1, 2), (2, 3), (3, 4), (4, 5), (5, 6).

(a) What is the domain of f?

(b) What is the range of f?

(c) What number is represented by $f(1)$, $f(4)$, $f(6)$, $-f(2)$, $f(3) + 1$.

(d) Define f by means of an expression in x.

19-3. Graphs of Functions.

One way to represent a function is by means of a graph. When a function f is defined, the graph of f is the graph of the truth set of the equation

$$y = f(x).$$

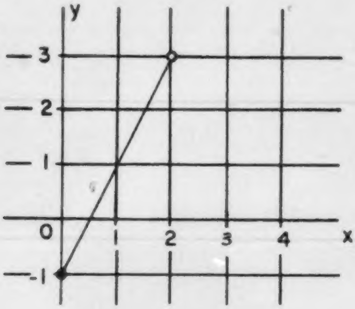
Example 1. Draw the graph of the function f defined by:

$$f(x) = 2x - 1, \quad 0 \leq x < 2.$$

You will note that the domain has been specified on the right. Do you see that D is the set of all real numbers greater than or equal to zero and less than 2? Thus 2 is not included in the domain, but 0 is. Our graph is the graph of the equation

$$y = 2x - 1, \quad 0 \leq x < 2.$$

Here for the first time we have an example of a graph of a line in the plane which has end points. It does not continue without end. You will note that the point $(2, 3)$ is circled, while the point $(0, -1)$ is filled in. You learned what this means on the number line in Chapter 1.



Is this the same as the graph of the function F defined by

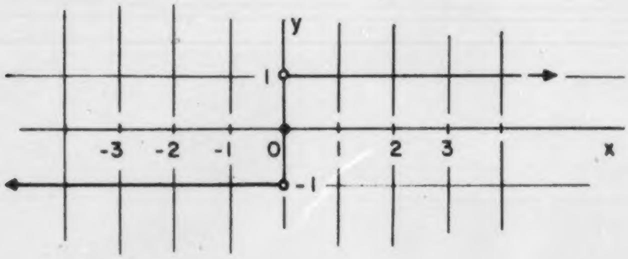
$$F(x) = 2x - 1, \quad -2 < x < 2?$$

What makes the difference?

Example 2. Draw the graph of the function g defined by:

$$g(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

Here the function is defined in three parts. As we said before, it is to be considered as a single function. The graph is also in three parts. Check the following drawing. Can you see that it is the graph of the equation $y = g(x)$?



Check Your Reading

1. If f is a function that is described in terms of the variable x, then the graph of f is the graph of what open sentence?
2. In the graph of f defined by $f(x) = 2x - 1$, $0 \le x < 2$, why is the point $(2, 3)$ circled? Why is the point $(0, -1)$ filled in?
3. In the graph of g defined in the section, why is the point $(0, -1)$ circled? Why is the point $(0, 1)$ circled? Why is the point $(0, 0)$ filled in?

Problem Set 19-3a

1. Draw the graphs of the functions defined as follows:
   (a) $f(x) = 3x$, $-3 \le x < 2$
   (b) $F(t) = 2t - 5$, $0 \le t \le 2$
   (c) $g(s) = 2s$, s an integer such that $-3 < s \le 5$
   (d) $T(s) = \frac{1}{3}s + 1$, $-1 \le s \le 2$

Problem Set 19-3a
(continued)

(e) $G(x) = |x|, \quad -3 \leq x \leq 3$

(f) $u(x) = \begin{cases} -x, & -3 \leq x < 0 \\ x, & 0 \leq x < 3 \end{cases}$

2. Draw the graphs of the functions defined as follows:

(a) $f(x) = x^2, \quad -3 < x < 3$

(b) $F(t) = t^2$, t an integer such that $-3 \leq t \leq 3$

(c) $V(t) = t^2 - 1, \quad -2 < t \leq 1$

(d) $h(x) = \sqrt{x}$, domain: $\{1, 4, 9, 16\}$

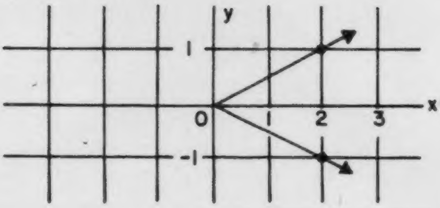
(e) $H(t) = \begin{cases} 4, & 4 < t \\ t^2, & 0 < t \leq 2 \\ 0, & t \leq 0 \end{cases}$

(f) $q(x) = \begin{cases} -1, & -5 \leq x < -1 \\ x, & -1 \leq x < 1 \\ x^2, & 1 < x \leq 2 \end{cases}$

3. What are the domains of definition and the ranges of the functions in Problem 1?

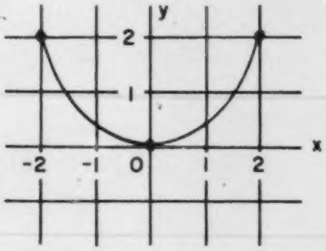
4. What are the domains of definition and the ranges of the functions in Problem 2?

Now that we know how to draw the graph of a function, it is natural to ask whether a given set of points in the plane is the graph of some function. You recall that in order to have a function we must have a rule which assigns to each element of the domain exactly one number. What does this mean in connection with a graph? Let us examine the following drawing.

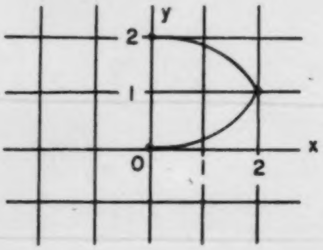


Do you see that there are two dots, one directly above and one directly below the coordinate 2 on the x-axis? This means that two different ordinates, 1 and -1, are associated with the abscissa 2. This shows that to the element 2 in the domain there have been assigned two different numbers. Thus we see that this is not the graph of a function. Why?

Examine the following two drawings. Do you see that the drawing marked A does represent a function, but the drawing marked B does not?



(A)



(B)

We can sum this up by asking a question. If a vertical line is drawn through the graph of a function, in how many points should the line intersect the graph? Do you see why the answer is one?

Check Your Reading

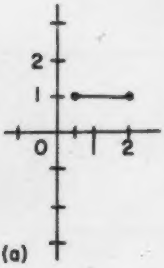
1. If the point (2, 1) is on the graph of a function f, why is it that (2, -1) cannot also be a point on the graph of f?
2. Why is it that drawing B in the text does not represent a function?

Check Your Reading
(continued)

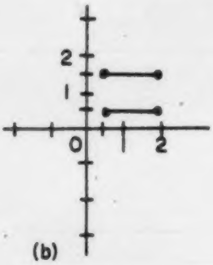
3. Describe a simple geometric test for deciding whether a graph is the graph of a function.

Problem Set 19-3b

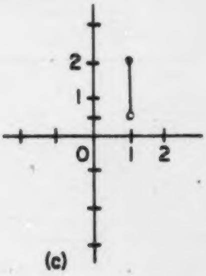
1. Consider the sets of points, coordinate axes not included, indicated in the following figures.



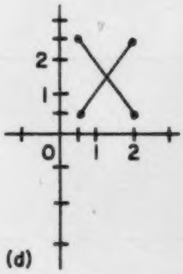
(a)



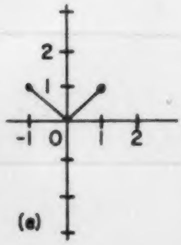
(b)



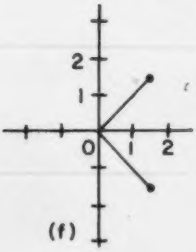
(c)



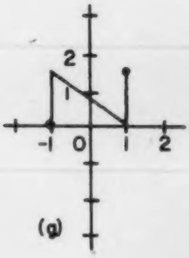
(d)



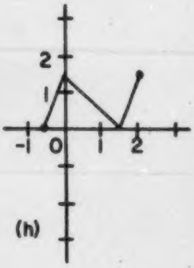
(e)



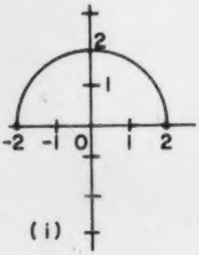
(f)



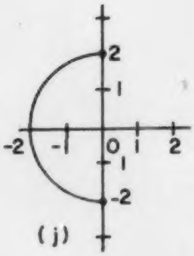
(g)



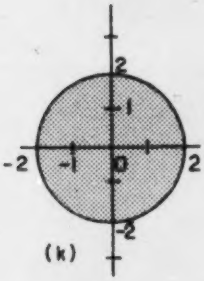
(h)



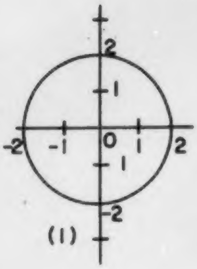
(i)



(j)



(k)



(l)

State which of the above figures is not the graph of some function, and in each case give your reasons.

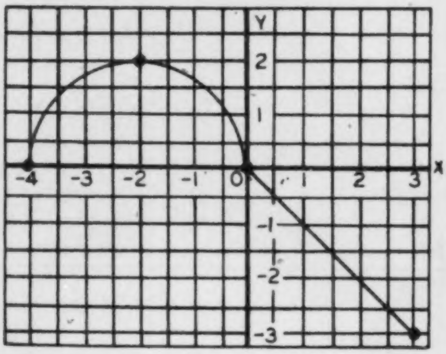
Problem Set 19-3b
(continued)

2. The accompanying figure is the graph of a function h. From the graph estimate

   (a) h(-3), h(0), h(2);

   (b) the domain of definition of h;

   (c) the range of h.



*3. Let G denote a set of points in the plane which is the graph of some function g.

   (a) For each x in the domain of definition of g, explain how to use the graph to obtain g(x).

   (b) How do you obtain the domain of definition of g from the graph of G?

   (c) Show that if (a, b) and (c, d) are any two distinct points of the graph G, then $a \neq c$.

Summary

The central idea in this chapter has been the concept of function. The description follows.

> Given a set of numbers and a rule which assigns to each number of this set exactly one number, the resulting association of numbers is called a function.
>
> The given set is called the domain of definition of the function, and the set of assigned numbers is called the range of the function.

We use letters, f, g, h, etc., to name a function. The symbol $f(x)$ is used to represent the value which the function f assigns to the number x, provided that x is a number in the domain of the function. For example, if $f(x) = 5x + 3$, for x a positive integer, then $f(2) = 13$, $f(4) = 23$, but $f(-3)$ and $f(\frac{1}{2})$ in this case have no meaning.

The graph of a function f is the graph of the open sentence

$$y = f(x).$$

If any vertical line meets a particular graph in more than one point, then this graph is not the graph of a function.

Review Problem Set

1. Which of the following sets of ordered pairs cannot be used to define a function?

   (a) (1, 2), (2, 3), (3, -3), (4, 0)
   (b) (1, 7), (1, 8), (2, 9), (2, 10)
   (c) (1, 1), (2, 4), (-2, 4), (3, 9), (-3, 9)
   (d) (1, 4), (2, 4), (3, 4), (5, 4), (6, 4)

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Review Problem Set
(continued)

(e) (3, 1), (3, 2), (3, 3), (3, 4), (3, 5)

(f) (1, 0), (2, 1), (3, 0), (4, 1), (5, 0)

2. Draw the graph of the line whose equation is $y = 3x$, for $-4 \leq x < 4$.

   (a) What is the y-intercept?

   (b) What is the slope of the line?

   (c) The graph you have drawn represents a function f. What numbers are represented by $f(-1)$, $f(0)$, $f(\frac{3}{2})$, $f(4)$?

   (d) Describe the graph of "$y = 3x + 2$".

   (e) Describe the graph of "$3y = 9x$".

   (f) g is a function defined by "$g(x) = 3x$", where x is an integer such that $-4 \leq x < 4$. Draw the graph of g. Is g the same function as f (part c) ?

3. H is a function described by the rule "to each real number assign its opposite".

   (a) What is the domain of H?

   (b) What is the range of H?

   (c) What number is represented by $H(3)$, $H(-3)$, $H(0)$, $H(1.75)$, $H(\sqrt{2})$, $H(x)$?

   (d) Write five ordered pairs which would be points on the graph of H.

   (e) What point, if any, will the graph of H have in common with the graph of "$y = 3x - 8$"?

   (f) If the point (a, b) is on the graph of H, which of the following points is also on H: (-a, b), (-a, -b), (a, -b)?

4. Suppose that the graph of "$y = x^2$" for $-3 \leq x \leq 3$ were given.

   (a) Explain how to obtain the graph of "$y = -x^2$" from the graph of "$y = x^2$".

   (b) Explain how to obtain the graph of "$y = x^2 + 2$" from the graph of "$y = x^2$".

   (c) Explain how to obtain the graph of "$y = (x + 1)^2$" from the graph of "$y = x^2$".

   (d) Why is the graph of "$y = x^2$" the graph of a function?

Review Problem Set
(continued)

(e) If h is a function defined by "$y = x^2$" $(-3 \leq x \leq 3)$, what is the range of h?

(f) What numbers do the following represent: $h(0)$, $h(-\sqrt{2})$, $h(\frac{3}{2})$, $h(4)$?

5. Find the truth sets of the following systems:

(a) $\begin{cases} x - 2y = 7 \\ 3x + y = 7 \end{cases}$

(b) $\begin{cases} 2m + n = -3 \\ 5m - n = -4 \end{cases}$

(c) $\begin{cases} 7k - 38 = 3\ell \\ k + 3\ell = 2 \end{cases}$

6. The ordered pair $(1, 3)$ is a solution of the system

$$\begin{cases} x - y + 2 = 0 \\ x + y - 4 = 0 \end{cases}$$

(a) Why is $(1, 3)$ the only solution? (Answer in terms of the graph of the system.)

(b) Why is $(1, 3)$ a solution of the system

$$\begin{cases} x - y + 2 = 0 \\ (x - y + 2) + (x + y - 4) = 0 \end{cases} ?$$

(c) Is the graph of the system in part (b) the same as the graph of the given system?

7. Given the open sentence

$$\frac{x}{x - 1} + \frac{3}{x} = \frac{37}{20}$$

(a) What is the domain of the variable?

(b) What is the least common multiple of $x - 1$, x, and 20?

(c) Why is $20x^2 + 60(x - 1) = 37(x - 1)x$ equivalent to the given open sentence?

(d) What is the truth set of the given open sentence?

Review Problem Set
(continued)

8. In each of the following, describe (if possible) the function in two ways: (i) by a table, (ii) by an expression in x. In each case, describe the domain of definition.

   (a) To each positive real number assign the product of $\frac{1}{3}$ and two more than the number.

   (b) With each positive integer n associate the n^{th} prime.

   (c) Associate with the number of dollars invested at 6% for one year the number of dollars earned as interest.

   (d) Associate with each length of the diameter of a circle the length of the circumference.

9. John drove 336 miles to Chicago, bought a new car and returned the next day by the same route. The return trip took 1 hour longer than the original trip and his average speed was 6 miles per hour slower. Find his average speed each way.

10. The sum of a number and its reciprocal is 4. Find the number.

11. In certain applications, the domain of definition of a function may be automatically restricted to those numbers which lead to meaningful results in the problem. For example, the area A of a rectangle with fixed perimeter 10 is given by $A = s(5 - s)$, where s is the length of a side in feet. The expression $s(5 - s)$ defines a function for all real s, but in this problem we must restrict s to numbers between 0 and 5. (Why?) What are the domains of definition of the functions involved in the following problems?

    (a) What amount of interest is earned by investing x dollars for a year at 4%?

Review Problem Set
(continued)

(b) A triangle has area 12 square inches, and its base measures x inches. What is the length of its altitude?

(c) An open top rectangular box is to be made by cutting a square of side x inches from each corner of a rectangular piece of tin measuring 10" by 8" and then folding up the sides. What is the volume of the box?

12. Consider the function k defined by:

$$k(x) = \begin{cases} \frac{x}{|x|}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

and the function g defined by:

$$g(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

Show that k is the same function as the function g.

13. Given the function H defined by:

$H(z) = z^2 - 1, \quad -3 < z < 3.$

Find the real numbers represented by:

(a) $H(2)$
(b) $H(\frac{1}{3})$
(c) $H(-\frac{1}{3})$
(d) $-H(-2)$
(e) $H(-1) + 1$
(f) $H(3)$
(g) $H(a)$
(h) $H(t - 1)$
(i) $H(t) - 1$

14. Draw the graphs of the functions defined as follows.

(a) $T(s) = \frac{1}{3}s + 1, \quad -1 \leq s \leq 2$

(b) $G(x) = |x|, \quad -3 \leq x \leq 3$

Review Problem Set
(continued)

(c) $U(x) = \begin{cases} -x, & -3 \leq x < 0 \\ x, & 0 \leq x < 3 \end{cases}$

(d) $V(t) = t^2 - 1, \quad -2 < t \leq 1$

(e) $h(x) = \begin{cases} -1, & x < 0 \\ 1, & x > 0 \end{cases}$

(f) $H(z) = \frac{|z|}{z}$

15. What are the domains of definition and the ranges of the functions defined in Problem 14?

16. Draw the graph of the function q defined by:

$$q(x) = \begin{cases} -1, & -5 \leq x < -1 \\ x, & -1 \leq x < 1 \\ x^2, & 1 < x \leq 2 \end{cases}$$

17. Give a rule for the definition of the function whose graph is the line extending from $(-2, 2)$ to $(4, -1)$, including end points.

18. Give a rule for the definition of the function whose graph consists of two line segments, one extending from $(-1, 1)$ to $(0, 0)$ with end points included, and the other extending from $(0, 0)$ to $(2, 1)$ with end points excluded. What are the domain of definition and range of this function?

19. Draw the graph of a function f which satisfies all of the following conditions over the domain of definition, $-2 \leq x \leq 2$:

$$\begin{aligned} f(-1) &= 2, \\ f(0) &= 0, \\ f(1) &= 0, \\ f(2) &= 2, \\ f(x) &< 0 \text{ for } 0 < x < 1. \end{aligned}$$

Review Problem Set
(continued)

20. Let f and g be functions defined by

$$f(x) = x^2 + x \qquad \text{and} \qquad g(x) = 2x + 6.$$

Find x such that $f(x) = g(x)$.

21. Draw the graph of the equation $y^2 = x$, for $0 \leq x < 4$. Is this the graph of some function?

22. Let f be a function defined by $f(x) = Ax + B$, for every x in the domain of f, where A and B are real numbers.

    (a) Describe the graph of f if $A = 0$.

    (b) Describe the graph of f if $A = 0$ and $B = 0$.

    (c) Determine A and B if the graph of f is the line segment joining $(-3, 0)$ and $(1, 2)$, including end points.

    (d) What is the domain of definition of the function in part (c) ?

    (e) Determine A and B if the graph of f is the line joining $(-1, 1)$ and $(3, 3)$ including end points.

    (f) What is the slope and y-intercept of the graph of the function in part (e)?

    (g) What is the domain of definition of the function in part (e)?

23. If L is the complete line containing the two points $(-3, 1)$ and $(1, -1)$, define the function h whose graph consists of the points (x, y) of L such that

$$-2 < y < 2.$$

24. Which of the following expressions define a function having a line as its graph?

    (a) $-(x - 2)$

    (b) $|x - 2|$

    (c) $\frac{1}{x - 2}$

    (d) $|x| - 2$

    (e) $(-x) - 2$

    (f) $x^2 - 2$

Review Problem Set
(continued)

25. Let f be the function defined by:

$$f(x) = x - 2, \text{ for every real number } x.$$

Write each expression in Problem 24 as a function g in terms of the given function f. Example: The expression (a) defines a function g such that

$$g(x) = -f(x), \text{ for every real number } x.$$

26. How are the graphs of f and g related in Problem 25, part (a)? In part (e)? Draw each graph with reference to a separate set of coordinate axes.

27. If F and G are functions defined for every real x by

$$F(x) = -3x + 2, \quad G(x) = 2x - 3,$$

explain how the graph of the sentence

$$\big(y - F(x)\big)\big(y - G(x)\big) = 0$$

is related to the graphs of F and G. (Do this without drawing the graphs of F and G.)

28. Draw the graphs of the following polynomials:

(a) $3x^2$

(b) $3x^2 + 3$

(c) $3(x - 3)^2$

(d) $3x(x - 3)$

(e) Explain how the graph of (d) can be obtained from the graph of (a).

29. Given the graph of $y = x^2$.

(a) Write an equation of the graph obtained by rotating the graph of $y = x^2$ one-half a revolution about the x-axis.

(b) Write the equation of the graph obtained by moving the graph of $y = x^2$ vertically upward 3 units.

(c) Write the equation of the graph obtained by moving the graph of $y = x^2$ horizontally 2 units to the left.

(d) Write the equation of the graph obtained by moving the graph of $y = x^2$ one unit to the right and two units down.

30. Draw graphs of the following open sentences.

(a) $y < x^2 + 6x + 5$

(b) $y = 4$ and $y = 3x^2 - 12x + 13$

(c) $y > 3x - 2x^2$

(d) $y = x^2 - 6|x| + 5$

31. Find the truth set of each of the following compound open sentences. Draw the graph of each clause. Does this help you with (b) and (c) ?

(a) $x - 2y + 6 = 0$ and $2x + 3y + 5 = 0$

(b) $2x - y - 5 = 0$ and $4x - 2y - 10 = 0$

(c) $2x + y - 4 = 0$ and $2x + y - 2 = 0$

32. Find the equation of the line through the intersection of the lines whose equations are $5x - 7y - 3 = 0$ and $3x - 6y + 5 = 0$ and passing through the origin. (Hint: What is the value of C so that the graph of $Ax + By + C = 0$ is a line through the origin?)

33. Find the truth set of each of the following systems:

(a) $\begin{cases} x - 4y - 15 = 0 \\ 3x + 5y - 11 = 0 \end{cases}$

(b) $\begin{cases} 2x = 3 - 2y \\ 3y = 4 - 2x \end{cases}$

(c) $\begin{cases} 2x = 3 - 2y \\ 3y = 4 - 2y \end{cases}$

(d) $\begin{cases} 2x = 3 - 2y \\ 3y = 4 - 3x \end{cases}$

(e) $\begin{cases} 3x + 2y = 1 \\ 2x - 3y = 18 \end{cases}$

(f) $\begin{cases} x - 2y = 0 \\ x + 2y = 0 \end{cases}$

(g) $\begin{cases} x = 2y - \frac{1}{6} \\ 2x + y = \frac{4}{3} \end{cases}$

(h) $\begin{cases} 3x - 4y - 1 = 0 \\ 7x + 4y - 9 = 0 \end{cases}$

(i) $\begin{cases} \frac{x}{2} - \frac{x}{3} = 1 \\ x + y = 7 \end{cases}$

(j) $\begin{cases} 6y + (2 - 4x) = 3 \\ 4x - 2(3y - 1) = 2 \end{cases}$

Review Problem Set
(continued)

34. Find two numbers whose sum is 56 and whose difference is 18.

35. The sum of Sally's and Joe's ages is 30 years. In five years the difference of their ages will be 4 years. What are their ages now?

36. A dealer has some cashew nuts that sell at $1.20 a pound and almonds that sell at $1.50 a pound. How many pounds of each should he put into a mixture of 200 pounds to sell at $1.32 a pound?

37. In a certain two digit number the units' digit is one more than twice the tens' digit. If the units' digit is added to the number, the sum is 35 more than three times the tens' digit. Find the number.

38. Hugh weighs 80 pounds and Fred weighs 100 pounds. They balance on a teeterboard that is 9 feet long. Each sits at an end of the board. How far is each boy from the point of balance?

39. Two boys sit on a see-saw, one five feet from the fulcrum (the point where it balances), the other on the other side six feet from the fulcrum. If the sum of the boys' weights is 209 pounds, how much does each boy weigh?

40. It takes a boat $1\frac{1}{2}$ hours to go 12 miles down stream, and 6 hours to return. Find the speed of the current and the speed of the boat in still water.

41. Three pounds of apples and four pounds of bananas cost $1.08, while 4 pounds of apples and 3 pounds of bananas cost $1.02. What is the price per pound of apples? Of bananas?

42. A and B are 30 miles apart. If they leave at the same time and walk in the same direction, A overtakes B in 60 hours. If they walk toward each other they meet in 5 hours. What are their speeds?

Review Problem Set
(continued)

43. A 90% solution of alcohol is to be mixed with a 75% solution to make 20 quarts of a 78% solution. How many quarts of the 90% solution should be used?

44. In a 300 mile race the driver of car A gives the driver of car B a start of 25 miles, and still finishes one-half hour sooner. In a second trial, the driver of car A gives the driver of car B a start of 60 miles and loses by 12 minutes. What were the average speeds of cars A and B in miles per hour?

45. Three hundred eleven tickets were sold for a basketball game, some for pupils and some for adults. Pupil tickets sold for 25 cents each and adult tickets for 75 cents each. The total money received was $108.75. How many pupil and how many adult tickets were sold?

46. The Boxer family is coming to visit, and no one knows how many children they have. Elsie, one of the girls, says she has as many brothers as sisters; her brother Jimmie says he has twice as many sisters as brothers. How many boys and how many girls are there in the Boxer family?

47. A home room bought three-cent and four-cent stamps to mail bulletins to the parents. The total cost was $12.67. If they bought 352 stamps, how many of each kind were there?

48. A bank teller has 154 bills of one-dollar and five-dollar denominations. He thinks his total is $465. Has he counted his money correctly?

49. The perimeter of a rectangle is 94 feet, and its area is 496 square feet. What are its dimensions?

50. An open box is constructed from a rectangular sheet of metal 8 inches longer than it is wide as follows: out of each corner a square of side 2 inches is cut, and the sides are folded up. The volume of the resulting box is 256 cubic inches. What were the dimensions of the original sheet of metal?

Review Problem Set
(continued)

51. A leg of a right triangle is 1 foot longer than the other leg and 8 feet shorter than the hypotenuse. Find the length of the sides of the right triangle.

52. A rope hangs from the window of a building. If pulled taut vertically to the base of the building there is 8 feet of rope lying slack on the ground. If pulled out taut until the end of the rope just reaches the ground, it reaches the ground at a point which is 28 feet from the building. How high above the ground is the window?

53. The hypotenuse of a right triangle is 3 units and the legs are equal in length. Find the length of a leg of the triangle.

54. Find the length of a diagonal of a square if the diagonal is 2 inches longer than a side.

55. The length of a rectangular piece of sheet metal is 3 feet more than the width. If the area is $46\frac{3}{4}$ square feet, find the length.

56. The sum of two numbers is 9. Find the numbers and their product if the product is the largest possible.

57. A boat manufacturer finds that his cost per boat in dollars is related to the number of boats manufactured each day by the formula,

$$c = n^2 - 10n + 175.$$

Find the number of boats he should manufacture each day so that his cost per boat is smallest.

58. The sum of two numbers is 9 and the difference of their squares is 25. Find the numbers.

59. The sum of 14 times a number and the square of the number is 11. Find the number.

CHALLENGE PROBLEMS

1. Let us think of moving all the points of a plane in the following manner: Each point with coordinates (c, d) is moved to the point with coordinates (-c, d). Describe this in terms of taking the opposite. Another way of looking at this is to consider that the points of the plane are rotated one-half a revolution about the y-axis, as indicated in the figure for the problem. Answer the following questions, and locate the points referred to in parts (a) and (b).

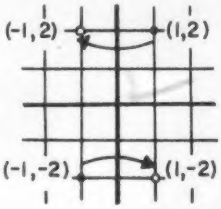

Figure for Problem 1

(a) To what points do the following points go:

(2, 1), (2, -1), $(-\frac{1}{2}, 2)$, (-1, -1), (3, 0),

(-5, 0), (0, 2), (0, -2) ?

(b) What points go to the points listed in (a) above?

(c) What point does (c, -d) go to?

(d) What point does (-c, d) go to?

(e) What point goes to (c, d) ?

(f) What points go to themselves?

2. Suppose a point with coordinates (c, d) is moved to the point (c + 2, d). This can be thought of as sliding the points of the plane to the right 2 units. Answer the following questions and locate all of the points in parts (a) and (b).

(a) What points do the following points go to:

(1, 1), (-1, 1), (-2, 2), (0, -3), (3, 0) ?

(b) What points go to the points listed in (a) above?

(c) To what point does (c - 2, d) go?

(d) What point goes to (-c, d) ?

(e) Which points go to themselves?

3. Draw a set of coordinate axes, designating them as the (x, y)-axes. Through point (2, -1) draw a pair of (a, b)-axes, making the a-axis parallel to the x-axis and the b-axis parallel to the y-axis. Locate the following points with reference to the (x, y)-axes: A(3, -5), B(-5, 3), C(-2, -5), D(0, 3), E(0, -3), F(-5, -1), G(-4, 3), H(6, 0), I(-6, 0), J(2, 6). Make a table giving the coordinates of each of these points with reference to the (a, b)-axes.

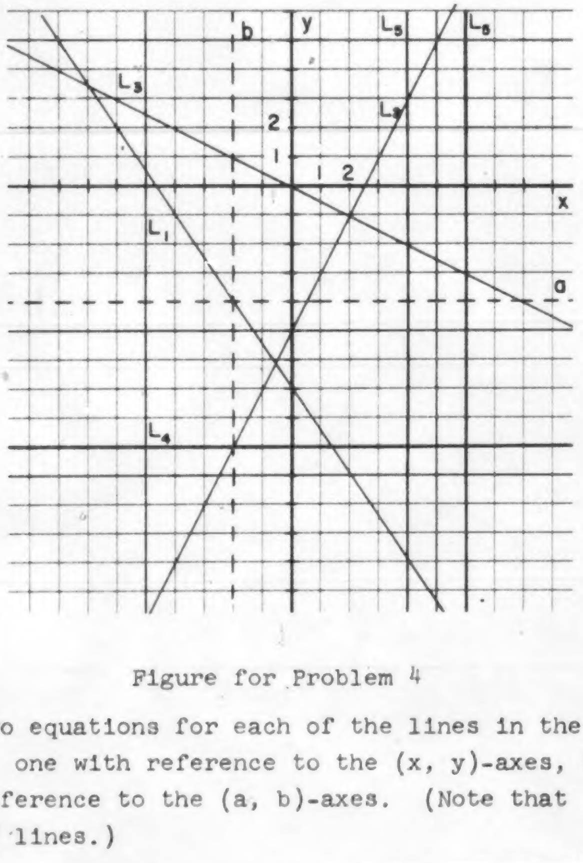

Figure for Problem 4

4. Give two equations for each of the lines in the above figure, one with reference to the (x, y)-axes, the other with reference to the (a, b)-axes. (Note that L_5 is a pair of lines.)

5. Suppose a point with coordinates (a, b) is moved to the point (a, -b). Describe this in terms of opposites. Describe it in terms of a rotation. Answer the following questions, and locate the points referred to in parts (a) and (b).

(a) What points do the following points go to?

$(2, 1)$, $(2, -1)$, $(-\frac{1}{2}, 2)$, $(-2, -3)$, $(3, 0)$, $(-5, 0)$, $(0, 5)$, $(0, -5)$.

(b) What points go to the points listed in (a)?
(c) What point does $(a, -b)$ go to?
(d) What point does $(-a, b)$ go to?
(e) What point goes to (a, b) ?
(f) What points go to themselves?

6. Suppose a point with coordinates (a, b) is moved to the point $(a - 3, b + 2)$. How can you obtain this by moving all the points of the plane? Answer the following questions and locate the points:

(a) What points do the following points go to?

$(1, 1)$, $(-1, -1)$, $(-2, 2)$, $(0, -3)$, $(3, 0)$?

(b) What points go to the above points?
(c) What point does $(a, b - 2)$ go to?
(d) What point goes to $(-a, -b)$?
(e) Which points go to themselves?
(f) Describe how the points are moved if (a, b) is moved to $(a, b - 2)$.

7. What does the graph of $|x| + |y| = 5$ look like? Let us make a chart first. Suppose we start with the intercepts. Let $y = 0$ and get some possible values of x which will make the sentence true. Then let $x = 0$ and get some values of y. Now fill in some of the other possible values.

8. Draw the graph of each of the following with reference to a separate set of axes.

(a) $|x| + |y| > 5$
(b) $|x| + |y| < 5$
(c) $|x| + |y| \leq 5$

9. Make a chart of some values which make the open sentence

$$|x| - |y| = 3$$

true, and draw the graph of the open sentence. Write four open sentences whose graphs form the same figure.

10. Draw the graph of "$y = 2|x|$". Give an equation of the graph which results from each of the following changes.

(a) The graph is rotated one-half revolution about the x-axis.
(b) The graph is moved 3 units to the right.
(c) The graph is moved 2 units to the left.
(d) The graph is moved 5 units up.
(e) The graph is moved 2 units to the right and 4 units down.

11. What is the slope of the line which contains the points $(-3, 2)$ and $(3, -4)$? If (x, y) is a point on this same line, verify that the slope is also $\frac{y - 2}{x - (-3)}$. Also verify that $\frac{y - (-4)}{x - 3}$ is the slope. If -1 and $\frac{y - 2}{x - (-3)}$ are different names for the slope, show that the equation of the line is "$y - 2 = (-1)(x + 3)$". Show that it can also be written "$y + 4 = (-1)(x - 3)$".

12. Write the equations of the lines through the following pairs of points. (Try to use the method of Problem 11 for parts (e) - (h).)

(a) $(0, 3)$ and $(-5, 2)$
(b) $(5, 8)$ and $(0, -4)$
(c) $(0, -2)$ and $(-3, -7)$
(d) $(5, -2)$ and $(0, 6)$
(e) $(-3, 3)$ and $(6, 0)$
(f) $(-3, 3)$ and $(-5, 3)$
(g) $(-3, 3)$ and $(-3, 5)$
(h) $(4, 2)$ and $(-3, 1)$

13. In the case of an expression of the form $\frac{k}{x}$, the value of the expression is said to vary inversely as the value of x. The number k is the constant of variation.

(a) Draw the graphs of the open sentences:

$$y = \frac{1}{x}; \quad y = -\frac{1}{x}; \quad y = \frac{2}{x}.$$

(b) If the variable x is given increasing positive values, what can you say of the values of $\frac{k}{x}$? Do they increase or decrease? (Does it matter whether k is positive or negative?)

14. A rectangle has an area of 25 square units, and one side has length w units.

   (a) Write an expression in terms of w for the length of the other side.

   (b) Is this a case of inverse variation? What is the constant of variation?

   (c) Draw the graph of the expression in (a).

15. Draw the graph of the truth set of each of these systems of inequalities. (The brace again indicates a compound sentence with connective and.)

   (a) $\begin{cases} x \geq 0 \\ y \geq 0 \\ 3x + 4y \leq 12 \end{cases}$ (c) $\begin{cases} -4 < x < 4 \\ -3 < y < 3 \end{cases}$

   (b) $\begin{cases} y \geq 2 \\ 4y \leq 3x + 8 \\ 4y + 5x \leq 40 \end{cases}$

In Problems 16 through 20, refer to Figure 1.

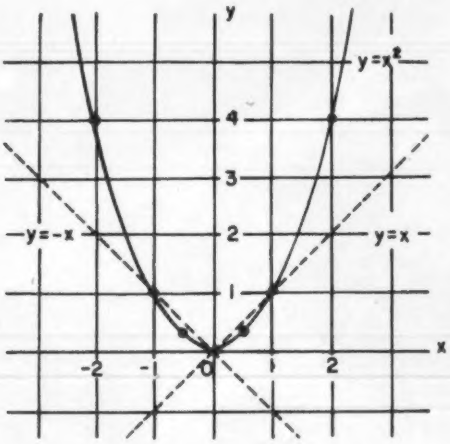

Figure 1

16. For any real number x, $x^2 \geq 0$. (Why?) Also, $x^2 = 0$ if and only if $x = 0$. Explain why the graph of $y = x^2$ lies entirely above the x-axis and touches the x-axis at a single point $(0, 0)$.

17. For any real number x,

$$(-x)^2 = x^2.$$

If (a, b) is a point on the graph of $y = x^2$, prove that $(-a, b)$ is also on the graph. (This means that the portion of the graph in Quadrant II can be obtained by rotating the portion in Quadrant I about the y-axis. We say that the graph of $y = x^2$ is symmetric about the y-axis.)

18. If x is any real number such that $0 < x < 1$, then $x^2 < x$. (Why?) Show that the portion of the graph of $y = x^2$, for $0 < x < 1$, lies below the graph of $y = x$.

19. If $1 < x$, then

$$x < x^2. \quad \text{(Why?)}$$

Show that the portion of the graph of $y = x^2$, for $1 < x$, lies above the line $y = x$.

20. If a and b are real numbers such that $0 < a < b$, then

$$a^2 < b^2. \quad \text{(Why?)}$$

Show that the graph of $y = x^2$ continues to rise as we move to the right from 0.

21. Show that a horizontal line will intersect the graph of $y = x^2$ in at most two points.

22. Choose any point (a, a^2) on the graph of $y = x^2$. What is the slope of the line containing $(0, 0)$ and (a, a^2)? As we choose points of the graph close to the origin (a close to 0) what happens to the slope of this line? Can you explain why the graph of $y = x^2$ is flat near the origin?

23. Prove the Theorem: Given any quadratic polynomial $Ax^2 + Bx + C$, there exist real numbers a, h, k such that $a(x - h)^2 + k = Ax^2 + Bx + C$, for every real number x. The numbers a, h, k are related to the numbers A, B, and C by the true sentences

$$a = A, \qquad h = -\frac{B}{2A}, \qquad k = \frac{4AC - B^2}{4A}.$$

24. The problem of changing a quadratic polynomial, such as $-2x^2 - 4x + 1$, into standard form can also be handled as follows. Let us find numbers a, h, k (if possible) such that

$$a(x - h)^2 + k = -2x^2 - 4x + 1,$$

for every real number x. By simplifying and regrouping the left member, we write

$$ax^2 - 2ahx + (ah^2 + k) = -2x^2 - 4x + 1,$$

for every real number x. Now we see at a glance that we must find a, h, k so that

$$a = -2, \qquad -2ah = -4, \qquad ah^2 + k = 1. \quad \text{(Why?)}$$

If $a = -2$, then "$-2ah = -4$" is equivalent to "$4h = -4$", i.e., to "$h = -1$". Also, if $a = -2$ and $h = -1$, then "$ah^2 + k = 1$" is equivalent to "$-2 + k = 1$", i.e., to "$k = 3$". With $a = -2$, $h = -1$, and $k = 3$, we have

$$-2(x + 1)^2 + 3 = -2x^2 - 4x + 1,$$

for every real number x.

Using this method write each of the following in standard form.

(a) $3x^2 - 7x + 5$

(b) $5x^2 - 3x + \frac{13}{20}$

(c) $Ax^2 + Bx + C$, where A, B, C are real numbers.

25. Consider the quadratic polynomial in standard form,

$a(x - h)^2 + k$, where a, h, k are real numbers and $a \neq 0$.

(a) State a rule for deciding whether or not this polynomial over the real numbers can be factored.

(b) If a, h, k are integers, what conditions on these numbers guarantee that this polynomial over the integers can be factored?

(c) State a rule for deciding whether the truth set of

$$a(x - h)^2 + k = 0$$

contains two, one, or no real numbers.

26. Consider the function Q defined by:

$$Q(x) = \begin{cases} -1, & -1 \leq x < 0 \\ x, & 0 < x \leq 2 \end{cases}$$

(a) What is the domain of definition of Q?

(b) What is the range of Q?

(c) What numbers are represented by

$$Q(-1), \quad Q(-\tfrac{1}{2}), \quad Q(0), \quad Q(\tfrac{1}{2}), \quad Q(\tfrac{3}{2}), \quad Q(\pi) \quad ?$$

(d) If R is defined by

$$R(z) = \begin{cases} z, & 0 < z \leq 2 \\ -1, & -1 \leq z < 0 \end{cases}$$

is R a different function from Q?

GLOSSARY CHAPTERS 16-19

EQUIVALENT SENTENCES - Open sentences which have the same truth set are called equivalent sentences.

EQUIVALENT SYSTEMS - Systems with the same truth set are called equivalent systems.

FUNCTION - Given a set of numbers and a rule which assigns to each number of this set exactly one number, the resulting association of numbers is called a function.

ORDERED NUMBER PAIR - A pair of numbers of the form $(2, 1)$ in which the first number represents a value of x and the second a value of y is called an ordered number pair.

QUADRATIC EQUATION - An open sentence of the form $Ax^2 + Bx + C = 0$, where A, B, and C are any real numbers with $A \neq 0$, is called a quadratic equation.

QUADRATIC POLYNOMIAL - A polynomial of the form $Ax^2 + Bx + C$, where A, B, and C are any real numbers with $A \neq 0$, is called a quadratic polynomial in x.

SLOPE - If a sentence is written in the form

$$y = ax + b$$

then a is the slope of the graph of the sentence. If A and B are two points on the graph of the sentence, then the slope of the graph is equal to the ratio:

$$\frac{\text{vertical distance from } A \text{ to } B}{\text{horizontal distance from } A \text{ to } B}.$$

SYSTEM OF SENTENCES - A compound open sentence with connecting word "and" is called a system of sentences. Such systems are often written in the following form.

$$\begin{cases} 2x - 3y = 5 \\ 3x + 5y = 4 \end{cases}$$

TRUTH SET OF A SYSTEM - The truth set of a system of sentences in x and y is the set of all ordered pairs (x, y) that satisfy all of the sentences of the system.

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